PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY.

Note to be inserted at p. 187, Part I. 1882.

CORRECTION TO A PAPER ENTITLED

"*On the Stresses caused in the Interior of the Earth by the Weight of Continents and Mountains.*"

Since this paper has left my hands I have discovered an error in the work. The error does not affect the physical conclusions, except in one unimportant respect; had it done so I should probably have found it out long ago.

Throughout the paper the normal stresses P, Q, R *require an additional term* W_i. *The only function of these stresses used for obtaining physical results is* P − R, *and it remains unchanged when the correction is made.* § 10 *must however be erased.*

The error takes its origin in § (1). Thomson's solution (1), when reduced to the form applicable to the incompressible solid, is the solution of the equations $-\frac{dp}{dx} + \upsilon\nabla\alpha^2 = \frac{dW}{dx}$, and two others. The solution required is that of $-\frac{dp}{dx} + \upsilon\nabla\alpha^2 = 0$, and two others. The W involved in my solution is not the potential of a true bodily force, but only an "effective potential" producing the same strains as those due to the weight of the continents and mountains, but causing a different hydrostatic pressure. When therefore p is determined from Thomson's solution, that p is really equal to $p + W_i$ of the problem of the continents. Hence equation (3) should be $p = -\left(1 + \frac{i}{I}\right)W_i$, instead of $p = -\frac{i}{I}W_i$. The correction to (3) must be carried on through the rest of the paper, and obviously it merely adds W_i to the stresses P, Q, R, leaving P − R unchanged.

The error would have been avoided had I, as suggested on p. 190, worked directly from the equations of equilibrium of the elastic incompressible solid, instead of from Thomson's solution.

When the solid is compressible, this method of "effective potential" [see "The Tides of a Viscous Spheroid," *Phil. Trans.* Part I. 1879, pp. 7—9] for including all the effects of gravitation, is not applicable without certain additional terms in α, β, γ. Hence § 10 is erroneous, inasmuch as the expressions for the strains and stresses are incomplete. The correction of § 10 (which is not difficult) would require too much space to be carried out in this note.

G. H. DARWIN.

Aug. 1, 1882.

IV. *On the Stresses caused in the Interior of the Earth by the Weight of Continents and Mountains.*

By G. H. DARWIN, *F.R.S.*

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[PLATES 19, 20.]

TABLE OF CONTENTS.

IN this paper I have considered the subject of the solidity and strength of the materials of which the earth is formed, from a point of view from which it does not seem to have been hitherto discussed.

The first part of the paper is entirely devoted to a mathematical investigation, based upon a well-known paper of Sir WILLIAM THOMSON'S. The second part consists of a summary and discussion of the preceding work. In this I have tried, as far as possible, to avoid mathematics, and I hope that a considerable part of it may prove intelligible to the non-mathematical reader.

I.

THE MATHEMATICAL INVESTIGATION.

§ 1. *On the state of internal stress of a strained elastic sphere.*

Let there be a homogeneous elastic sphere, for which $\omega-\frac{1}{3}\upsilon$ is the modulus of compressibility (or incompressibility, as I shall call it) and υ the rigidity.* Take the centre of the sphere as the origin for a set of rectangular axes x, y, z. Let the sphere be subjected to no surface stresses, let it be devoid of gravitation, but subject to internal force such that the force acting on a unit volume of the elastic solid is expressible by a gravitation potential W_i, a solid spherical harmonic of the i^{th} degree of the coordinates x, y, z.

Let w be the density of the elastic solid, a the radius of the sphere, and r the radius vector of any point measured from the centre of the sphere.

Sir WILLIAM THOMSON has investigated the state of internal strain produced under the conditions above described. If α, β, γ be the displacements his solution is as follows :—

$$\left.\begin{aligned} \alpha&=(E_i a^2-F_i r^2)\frac{dW_i}{dx}-G_i r^{2i+3}\frac{d}{dx}(W_i r^{-2i-1}) \\ \text{where}\quad E_i&=\frac{i[(i+2)\omega-\upsilon]}{2(i-1)\upsilon\{[2(i+1)^2+1]\omega-(2i+1)\upsilon\}} \\ F_i&=\frac{(i+1)(2i+3)\omega-(2i+1)\upsilon}{2(2i+1)\upsilon\{[2(i+1)^2+1]\omega-(2i+1)\upsilon\}} \\ G_i&=\frac{i\omega}{(2i+1)\upsilon\{[2(i+1)^2+1]\omega-(2i+1)\upsilon\}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and similar expressions for β and γ.†

Now let P, Q, R, S, T, U be the six stresses, across three planes mutually at right angles at the point x, y, z, estimated as is usual in works on the theory of elasticity.

Let P, Q, R be tractions and not pressures, and let p be the hydrostatic pressure at the point x, y, z.

Then P+Q+R being an invariant of the stress quadric, we have,

$$p=-\tfrac{1}{3}(\mathrm{P}+\mathrm{Q}+\mathrm{R})$$

if $\delta=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}$, so that δ is the dilatation, then according to the usual formulas,‡

* The phraseology adopted by THOMSON and TAIT (first edition) and others seems a little unfortunate. One might be inclined to suppose that compressibility and rigidity were things of the same nature; but rigidity and the reciprocal of compressibility are of the same kind. If one may give exact meanings to old words of somewhat general meanings, then one may pair together compressibility and "pliancy," and call the moduli for the two sorts of elasticity the "incompressibility" and rigidity.

† THOMSON and TAIT'S 'Nat. Phil.,' § 834, (8) and (9); or Phil. Trans., 1863, p. 573.

‡ THOMSON and TAIT'S 'Nat. Phil.,' § 693.

$$P=(\omega-v)\delta+2v\frac{d\alpha}{dx}$$

$$T=v\left(\frac{d\alpha}{dz}+\frac{d\gamma}{dx}\right)$$

and the other four stresses are expressed from these by cyclic changes of P, Q, R; S, T, U; α, β, γ; x, y, z.

The first task is to find p.

Now by adding P, Q, R together we have,

$$p=-(\omega-\tfrac{1}{3}v)\delta$$

We must now find δ from (1).

By differentiation

$$\frac{d\alpha}{dx}=(E_i\alpha^2-F_ir^2)\frac{d^2W_i}{dx^2}-2F_ix\frac{dW_i}{dx}-G_ir^{2i+3}\frac{d^2}{dx^2}(W_ir^{-2i-1})-(2i+3)G_ir^{2i+1}x\frac{d}{dx}(W_ir^{-2i-1})$$

and similar expressions for $d\beta/dy$, $d\gamma/dz$.

Now W_i, W_ir^{-2i-1} are spherical harmonics of degrees i, $-i-1$, and are also homogeneous functions of the same degrees. If therefore we add the three expressions together, and note the properties of harmonics and of homogeneous functions, we have

$$\delta=-2iF_iW_i+(2i+3)(i+1)G_iW_i$$

Omitting for brevity that part of the divisors in the expressions for F and G which is common to both,

$$\begin{aligned}-2iF_i+(2i+3)(i+1)G_i&=-i(i+1)(2i+3)\omega+i(2i+1)v+(2i+3)(i+1)i\omega\\&=\quad i(2i+1)v\end{aligned}$$

and we have, on introducing the omitted denominator,

$$\delta=\frac{i}{[2(i+1)^2+1]\omega-(2i+1)v}W_i$$

And

$$p=\frac{-i(\omega-\frac{1}{3}v)}{[2(i+1)^2+1]\omega-(2i+1)v}W_i$$

Throughout the rest of this paper (excepting in § 10) the elastic sphere will be treated as incompressible, so that ω is to be considered as infinitely large compared with v.

Henceforth I write

$$I=2(i+1)^2+1 \quad \ldots\ldots\ldots\ldots \quad (2)$$

and when ω is infinite compared with v, we have,

$$p=-\frac{i}{I}W_i \quad \ldots\ldots\ldots\ldots \quad (3)$$

Also we may put

$$\left.\begin{aligned} \mathrm{P}&=-p+2v\frac{d\alpha}{dx} \\ \mathrm{T}&=v\left(\frac{d\alpha}{dz}+\frac{d\gamma}{dx}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

And on putting ω infinite in (1) we have

$$\left.\begin{aligned} \alpha=\frac{1}{Iv}\left[\left\{\frac{i(i+2)}{2(i-1)}a^2-\frac{(i+1)(2i+3)}{2(2i+1)}r^2\right\}\frac{dW_i}{dx}-\frac{i}{2i+1}r^{2i+3}\frac{d}{dx}(W_ir^{-2i-1})\right] \\ \text{and symmetrical expressions for } \beta \text{ and } \gamma. \end{aligned}\right\} \quad . \quad . \quad (5)$$

The hydrostatic pressure might have been found from this general solution for the case of incompressibility, but in order to do so it would have been necessary to go back to the equations of equilibrium of the solid, and I prefer to deduce it from Sir WILLIAM THOMSON'S solution in the more general case.

Since

$$r^{2i+3}\frac{d}{dx}(W_ir^{-2i-1})=-(2i+1)xW_i+r^2\frac{dW_i}{dx}$$

and since

$$(i+1)(2i+3)+2i=(2i+1)(i+3)$$

we may write α as follows :—

$$\alpha=\frac{1}{2Iv}\left[\left\{\frac{i(i+2)}{i-1}a^2-(i+3)r^2\right\}\frac{dW_i}{dx}+2ixW_i\right] \quad . \quad . \quad . \quad . \quad (6)$$

In order to find the stresses P, Q, &c., we must now evaluate $\frac{d\alpha}{dx}$, $\frac{d\alpha}{dz}$, $\frac{d\gamma}{dx}$, &c.

Differentiating (6) with regard to x,—

$$2Iv\frac{d\alpha}{dx}=\left\{\frac{i(i+2)}{i-1}a^2-(i+3)r^2\right\}\frac{d^2W_i}{dx^2}-6x\frac{dW_i}{dx}+2iW_i \quad . \quad . \quad . \quad . \quad (7)$$

Differentiating with regard to z,—

$$2Iv\frac{d\alpha}{dz}=\left\{\frac{i(i+2)}{i-1}a^2-(i+3)r^2\right\}\frac{d^2W_i}{dxdz}+2\left\{ix\frac{dW_i}{dz}-(i+3)z\frac{dW_i}{dx}\right\} \quad . \quad . \quad . \quad (8)$$

and by symmetry

$$2Iv\frac{d\gamma}{dx}=\left\{\frac{i(i+2)}{i-1}a^2-(i+3)r^2\right\}\frac{d^2W_i}{dxdz}+2\left\{iz\frac{dW_i}{dx}-(i+3)x\frac{dW_i}{dz}\right\} \quad . \quad . \quad . \quad (9)$$

Adding (8) and (9) together and dividing by 2, we have

$$Iv\left(\frac{d\alpha}{dz}+\frac{d\gamma}{dx}\right)=\left\{\frac{i(i+2)}{i-1}a^2-(i+3)r^2\right\}\frac{d^2W_i}{dxdz}-3\left\{x\frac{dW_i}{dz}+z\frac{dW_i}{dx}\right\} \quad . \quad . \quad . \quad (10)$$

Hence from (3) (4) (7) and (10) we have,

$$
\left.\begin{aligned}
I\mathrm{P} &= \left\{\frac{i(i+2)}{i-1}a^2-(i+3)r^2\right\}\frac{d^2 W_i}{dx^2}-6x\frac{dW_i}{dx}+3i\,W_i \\
I\mathrm{T} &= \left\{\frac{i(i+2)}{i-1}a^2-(i+3)r^2\right\}\frac{d^2 W_i}{dxdz}-3\left(x\frac{dW_i}{dz}+z\frac{dW_i}{dx}\right)
\end{aligned}\right\} \quad . \; . \; . \; . \quad (14)
$$

where $I=2(i+1)^2+1$.

The expressions for Q, R, S, U may be written down from these by means of cyclic changes of the symbols.

These are the required expressions for the stresses at any point in the interior of the sphere.

In order to find the magnitude and direction of the principal stress-axes at any point it would be necessary to solve a cubic equation. The solution of this equation appears to be difficult, but the special case in which it reduces to a quadratic equation will fortunately give adequate results. It may be seen from considerations of symmetry that if W_i be a zonal harmonic, two of the principal stress-axes lie in a meridional plane and the third is perpendicular thereto. Moreover the greatest and least stress-axes are those which lie in that plane, and the mean stress-axis is that which is perpendicular thereto. If this is not obvious to the reader at present, it will become so later.

I shall therefore take W_i to be a zonal harmonic, and as the future developments will be by means of series (which though finite will be long for the higher orders of harmonics) I shall attend more especially to the equatorial regions of the sphere.

§ 2. *The determination of the stresses when the disturbing potential is an even zonal harmonic.*

If θ be colatitude the expression for a zonal surface harmonic or LEGENDRE'S function of order i is

$$
\cos^i\theta-\frac{i(i-1)}{4.(1\,!)^2}\cos^{i-2}\theta\sin^2\theta+\frac{i(i-1)(i-2)(i-3)}{4^2(2\,!)^2}\cos^{i-4}\theta\sin^4\theta-\ldots
$$

or if we begin by the other end of the series, and take i as an even number, the expression is

$$
(-)^{\frac{1}{2}i}\frac{i\,!}{2^i\{\frac{1}{2}i\,!\}^2}\left[\sin^i\theta-\frac{i^2}{2\,!}\sin^{i-2}\theta\cos^2\theta+\frac{i^2(i-2)^2}{4\,!}\sin^{i-4}\theta\cos^4\theta-\ldots\right]
$$

This latter is the appropriate form when we wish to consider especially the equatorial regions, because $\cos\theta$ is small for that part of the sphere.

There is of course a similar formula when i is odd, but of this I shall make no use.

Now let $\rho^2=y^2+z^2$, so that $\sin\theta=\rho/r$, $\cos\theta=z/r$.

Then we may put

$$W_i=\rho^i-\frac{i^2}{2!}\rho^{i-2}z^2+\frac{i^2(i-2)^2}{4!}\rho^{i-4}z^4-\frac{i^2(i-2)^2(i-4)^2}{6!}\rho^{i-6}z^6+\ldots \quad (12)$$

W_i is a solid zonal harmonic of degree i; but $r^{-i}W_i$ requires multiplication by a factor $(-)^{\frac{1}{2}i}i!/2^i\{\frac{1}{2}i!\}^2$ in order to make it a LEGENDRE'S function.

The factors by which W_i must be deemed to be multiplied in order that it may be a potential, will be dropped for the present, to be inserted later. Or we may, if we like, suppose that the units of length or of time are so chosen as to make the factor equal to unity.

Now let

$$\beta_0=1,\ \beta_2=\frac{i^2}{2!},\ \beta_4=\frac{i^2(i-2)^2}{4!},\ \beta_6=\frac{i^2(i-2)^2(i-4)^2}{6!}\ \&c. \quad (13)$$

Then, dropping the suffix to W for brevity, we may write

$$W=\beta_0\rho^i-\beta_2\rho^{i-2}z^2+\beta_4\rho^{i-4}z^4-\beta_6\rho^{i-6}z^6+\ldots \quad (14)$$

I shall now find P, Q, R, T at any point in the meridional plane which is determined by $y=0$.

In evaluating the first differential coefficients of W we must not put $y=0$, in as far as these coefficients are a first step towards the determination of the second differential coefficients. But in as far as these first coefficients are directly involved in the expressions for P, Q, R, and T, and in the second coefficients in the same expressions, we may put $y=0$, and thus write x in place of ρ.

$$\rho\frac{d\rho}{dx}=x,\ \rho\frac{d\rho}{dy}=y,\ \frac{d\rho}{dz}=0,\ \text{since } \rho^2=x^2+y^2.$$

Then

$$\frac{dW}{dx}=x[i\beta_0\rho^{i-2}-(i-2)\beta_2\rho^{i-4}z^2+(i-4)\beta_4\rho^{i-6}z^4-\ldots]$$

$$\frac{dW}{dy}=y[\text{same series}]$$

$$\frac{dW}{dz}=z[\quad -2\beta_2\rho^{i-2}+4\beta_4\rho^{i-4}z^2-6\beta_6\rho^{i-6}z^4+\ldots]$$

In differentiating a second time we may treat ρ as identical with x, because y is to be put equal to zero. Thus

$$\left.\begin{aligned}\frac{d^2W}{dx^2}&=i(i-1)\beta_0x^{i-2}-(i-2)(i-3)\beta_2x^{i-4}z^2+(i-4)(i-5)\beta_4x^{i-6}z^4-\ldots\\ \frac{d^2W}{dy^2}&=i\beta_0x^{i-2}-(i-2)\beta_2x^{i-4}z^2+(i-4)\beta_4x^{i-6}z^4-\ldots\\ \frac{d^2W}{dz^2}&=-1.2\beta_2x^{i-2}+3.4\beta_4x^{i-4}z^2-5.6\beta_6x^{i-6}z^4+\ldots\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}&\frac{d^2W}{dxdz}=xz[-2(i-2)\beta_2x^{i-4}+4(i-4)\beta_4x^{i-6}z^2-6(i-6)\beta_6x^{i-8}z^4+\ldots]\\&\frac{d^2W}{dxdy}=0,\qquad \frac{d^2W}{dydz}=0\end{aligned}\right\}\quad . \quad (16)$$

Also treating ρ as identical with x, and putting $y=0$,

$$\left.\begin{aligned}&x\frac{dW}{dx}=i\beta_0x^i-(i-2)\beta_2x^{i-2}z^2+(i-4)\beta_4x^{i-4}z^4-\ldots\\&y\frac{dW}{dy}=0\\&z\frac{dW}{dz}=-2\beta_2x^{i-2}z^2+4\beta_4x^{i-4}z^4-6\beta_6x^{i-6}z^6+\ldots\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (17)$$

$$\left.\begin{aligned}&\left(z\frac{dW}{dx}+x\frac{dW}{dz}\right)=xz\{(i\beta_0-2\beta_2)x^{i-2}-[(i-2)\beta_2-4\beta_4]x^{i-4}z^2\\&\qquad\qquad\qquad\qquad+[(i-4)\beta_4-6\beta_6]x^{i-6}z^4-\ldots\}\\&\left(y\frac{dW}{dx}+x\frac{dW}{dy}\right)=0,\qquad \left(y\frac{dW}{dz}+z\frac{dW}{dy}\right)=0\end{aligned}\right\}\quad . \quad (18)$$

These various results have now to be introduced into the expressions (11) for P, Q, R, S, T, U.

In performing these operations it will be convenient to write J for $i(i+2)/(i-1)$. Also $r^2=\rho^2+z^2=x^2+z^2$, when $y=0$.

From these formulas we see that S=0, U=0; which shows that a meridional plane is one of the three principal planes, a result already observed from principles of symmetry.

Now

$$\left.\begin{aligned}&r^2\frac{d^2W}{dx^2}=i(i-1)\beta_0x^i+[i(i-1)\beta_0-(i-2)(i-3)\beta_2]x^{i-2}z^2\\&\qquad\qquad-[(i-2)(i-3)\beta_2-(i-4)(i-5)\beta_4]x^{i-4}z^4+\ldots\\&r^2\frac{d^2W}{dy^2}=i\beta_0x^i+[i\beta_0-(i-2)\beta_2]x^{i-2}z^2-[(i-2)\beta_2-(i-4)\beta_4]x^{i-4}z^4+\ldots\\&r^2\frac{d^2W}{dz^2}=-1.2\beta_2x^i-[1.2\beta_2-3.4\beta_4]x^{i-2}z^2+[3.4\beta_4-5.6\beta_6]x^{i-4}z^4-\ldots\end{aligned}\right\}\quad (19)$$

$$\left.\begin{aligned}&-2x\frac{dW}{dx}+iW=-i\beta_0x^i+(i-4)\beta_2x^{i-2}z^2-(i-8)\beta_4x^{i-4}z^4+\ldots\\&-2y\frac{dW}{dy}+iW=i\beta_0x^i-i\beta_2x^{i-2}z^2+i\beta_4x^{i-4}z^4-\ldots\\&-2z\frac{dW}{dz}+iW=i\beta_0x^i-(i-4)\beta_2x^{i-2}z^2+(i-8)\beta_4x^{i-4}z^4-\ldots\end{aligned}\right\}\quad . \quad . \quad (20)$$

$$r^2\frac{d^2W}{dxdz}=xz\{-2(i-2)\beta_2x^{i-2}-[2(i-2)\beta_2-4(i-4)\beta_4]x^{i-4}z^2$$
$$+[4(i-4)\beta_4-6(i-6)\beta_6]x^{i-6}z^6-\ldots\} \quad . \quad . \quad (21)$$

Then multiplying (19) by $-(i+3)$, (20) by 3, and (15) by Ja^2, and adding them each to each, we get the expressions for P, Q, R.

Also multiplying (21) by $-(i+3)$, (18) by -3, and (16) by Ja^2 and adding, we get the expression for T. The results are

$$IP=-[(i+3)i(i-1)+3i]\beta_0x^i$$
$$+[\{(i+3)(i-2)(i-3)+3(i-4)\}\beta_2-(i+3)i(i-1)\beta_0]x^{i-2}z^2$$
$$-[\{(i+3)(i-4)(i-5)+3(i-8)\}\beta_4-(i+3)(i-2)(i-3)\beta_2]x^{i-4}z^4$$
$$+[\{i+3)(i-6)(i-7)+3(i-12)\}\beta_6-(i+3)(i-4)(i-5)\beta_4]x^{i-6}z^6-\ldots$$
$$+Ja^2[i(i-1)\beta_0x^{i-2}-(i-2)(i-3)\beta_2x^{i-4}z^2+(i-4)(i-5)\beta_4x^{i-6}z^4-\ldots]$$

$$IR=[(i+3).1.2\beta_2+3i\beta_0]x^i-[(i+3).3.4\beta_4-\{(i+3).1.2-3(i-4)\}\beta_2]x^{i-2}z^2$$
$$+[(i+3).5.6\beta_6-\{(i+3).3.4-3(i-8)\}\beta_4]x^{i-4}z^4$$
$$-[(i+3).7.8\beta_8-\{(i+3).5.6-3(i-12)\beta_6]x^{i-6}z^6+\ldots$$
$$-Ja^2[1.2\beta_2x^{i-2}-3.4\beta_4x^{i-4}z^2+5.6\beta_6x^{i-6}z^4-\ldots]$$

$$IQ=-[(i+3)i-3i]\beta_0x^i+[\{(i+3)(i-2)-3i\}\beta_2-(i+3)i\beta_0]x^{i-2}z^2$$
$$-[\{(i+3)(i-4)-3i\}\beta_4-(i+3)(i-2)\beta_2]x^{i-4}z^4$$
$$+[\{(i+3)(i-6)-3i\}\beta_6-(i+3)(i-4)\beta_4]x^{i-6}z^6-\ldots$$
$$+Ja^2[ix^{i-2}-(i-2)\beta_2x^{i-4}z^2+(i-4)\beta_4x^{i-6}z^4-\ldots]$$

$$\frac{IT}{xz}=[\{(i+3)2(i-2)+3.2\}\beta_2-3i\beta_0]x^{i-2}$$
$$-[\{(i+3)4(i-4)+3.4\}\beta_4-\{(i+3)2(i-2)+3(i-2)\}\beta_2]x^{i-4}z^2$$
$$+[\{(i+3)6(i-6)+3.6\}\beta_6-\{(i+3)4(i-4)+3(i-4)\}\beta_4]x^{i-6}z^4$$
$$-[\{(i+3)8(i-8)+3.8\}\beta_8-\{(i+3)6(i-6)+3(i-6)\}\beta_6]x^{i-6}z^6+\ldots$$
$$-Ja^2[2(i-2)\beta_2x^{i-4}-4(i-4)\beta_4x^{i-6}z^2+6(i-6)\beta_6x^{i-8}z^4-\ldots]$$

The general law of formation of the successive coefficients is obvious, and it is easy to write down the general term in each of the eight series involved in these four expressions; the best way indeed of obtaining the formulas given below is to write down and transform the general term.

The semi-polar coordinates used hitherto are not so convenient as true polar coordinates; I therefore substitute r, radius vector, and l, latitude, for the x, z system, and putting $x=r\cos l$, $z=r\sin l$ write

$$\left.\begin{aligned}
\mathrm{P}&=r^i\cos^i l(A_0+A_2\tan^2 l+A_4\tan^4 l+\ldots)\\
&\qquad+a^2r^{i-2}\cos^{i-2}l(B_0+B_2\tan^2 l+B_4\tan^4 l+\ldots)\\
\mathrm{R}&=r^i\cos^i l(C_0+C_2\tan^2 l+C_4\tan^4 l+\ldots)\\
&\qquad+a^2r^{i-2}\cos^{i-2}l(D_0+D_2\tan^2 l+D_4\tan^4 l+\ldots)\\
\mathrm{T}&=r^i\sin l\cos^{i-1}l(E_0+E_2\tan^2 l+E_4\tan^4 l+\ldots)\\
&\qquad+a^2r^{i-2}\sin l\cos^{i-3}l(F_0+F_2\tan^2 l+F_4\tan^4 l+\ldots)\\
\mathrm{Q}&=r^i\cos^i l(G_0+G_2\tan^2 l+G_4\tan^4 l+\ldots)\\
&\qquad+a^2r^{i-2}\cos^{i-2}l(H_0+H_2\tan^2 l+H_4\tan^4 l+\ldots)
\end{aligned}\right\}\quad(22)$$

Then introducing for J and for the β's their values in terms of i, I find that the coefficients A, B, &c., are reducible to the forms given in the following equations:—

$$\left.\begin{aligned}
IA_0&=-\frac{1}{0!}\{i(i+2)(i-0)-3.0.(i+1)\}+0.(i+3)(i+2)(i+1)=-i^2(i+2)\\
IA_2&=\frac{i^2}{2!}\{i(i-0)(i-2)-3.2(i-1)\}-\frac{1}{0!}(i+3)(i+0)(i-1)\\
IA_4&=-\frac{i^2(i-2)^2}{4!}\{i(i-2)(i-4)-3.4(i-3)\}+\frac{i^2}{2!}(i+3)(i-2)(i-3)\\
IA_6&=\frac{i^2(i-2)^2(i-6)^2}{6!}\{i(i-4)(i-6)-3.6(i-5)\}-\frac{i^2(i-2)^2}{4!}(i+3)(i-4)(i-5)\\
\&c.&=\&c.\\
IB_0&=\frac{i(i+2)}{i-1}\frac{1}{0!}i(i-1)\\
IB_2&=-\frac{i(i+2)}{i-1}\frac{i^2}{2!}(i-2)(i-3)\\
IB_4&=\frac{i(i+2)}{i-1}\frac{i^2(i-2)^2}{4!}(i-4)(i-5)\\
\&c.&=\&c.
\end{aligned}\right\}\quad(23)$$

$$\left.\begin{aligned}
IC_0&=\frac{1}{0!}\{(i+3)i^2-[i(1(-2)-1)+3.0.1]\}=i[(i+1)(i+2)+1]\\
IC_2&=-\frac{i^2}{2!}\{(i+3)(i-2)^2-[i(3.0-1)+3.2.3]\}\\
IC_4&=\frac{i^2(i-2)^2}{4!}\{(i+3)(i-4)^2-[i(5.2-1)+3.4.5]\}\\
IC_6&=-\frac{i^2(i-2)^2(i-4)^2}{6!}\{(i+3)(i-6)^2-[i(7.4-1)+3.6.7]\}\\
\&c.&=\&c.\\
ID_0&=-\frac{i(i+2)}{i-1}\frac{i^2}{0!}\\
ID_2&=\frac{i(i+2)}{i-1}\frac{i^2(i-2)^2}{2!}\\
ID_4&=-\frac{i(i+2)}{i-1}\frac{i^2(i-2)^2(i-4)^2}{4!}\\
\&c.&=\&c.
\end{aligned}\right\}\quad(24)$$

$$\left.\begin{aligned}
IE_0&=-\frac{1}{1!}i[1(3i+3)-i(i-0)(i+1)]=i(i+1)(i^2-3)\\
IE_2&=\frac{i^2}{3!}(i-2)[3(5i+3)-i(i-2)(i-1)]\\
IE_4&=-\frac{i^2(i-2)^2}{5!}(i-4)[5(7i+3)-i(i-4)(i-3)]\\
IE_6&=\frac{i^2(i-2)^2(i-4)^2}{7!}(i-6)[7(9i+3)-i(i-6)(i-5)]\\
\&c.&=\&c.\\
IF_0&=-\frac{i^3(i^2-4)}{i-1}\\
IF_2&=\frac{i^3(i^2-4)}{i-1}\,\frac{(i-2)(i-4)}{3!}\\
IF_4&=-\frac{i^3(i^2-4)}{i-1}\,\frac{(i-2)(i-4)^2(i-6)}{5!}\\
IF_6&=\frac{i^3(i^2-4)}{i-1}\,\frac{(i-2)(i-4)^2(i-6)^2(i-8)}{7!}\\
\&c.&=\&c.
\end{aligned}\right\}\quad .\ .\ .\ .\ (25)$$

$$\left.\begin{aligned}
IG_0&=-\frac{1}{0!}\{i(i-0)-3.0\}+0.(i+3)(i+2)=-i^2\\
IG_2&=\frac{i^2}{2!}\{i(i-2)-3.2\}-\frac{1}{0!}(i+3)i\\
IG_4&=-\frac{i^2(i-2)^2}{4!}\{i(i-4)-3.4\}+\frac{i^2}{2!}(i+3)(i-2)\\
\&c.&=\&c.\\
IH_0&=\frac{i(i+2)}{i-1}\,\frac{1}{0!}i\\
IH_2&=-\frac{i(i+2)}{i-1}\,\frac{i^2}{2!}(i-2)\\
IH_4&=\frac{i(i+2)}{i-1}\,\frac{i^2(i-2)^2}{4!}(i-4)\\
\&c.&=\&c.
\end{aligned}\right\}\quad .\ .\ .\ .\ (26)$$

These sets of coefficients are all written down in such a form that the laws of their formation are obvious, and the general terms may easily be found. I have computed their values from these formulas for the even zonal harmonics of orders 2, 4, 6, 8, 10, 12; the results are given in the following tables both in the form of fractions and of decimals approximately equal to those fractions.

The G's and H's were not computed because their values were not required for subsequent operations.

TABLE I.—The coefficients for expressing the stress P.

i	A_0	A_2	A_4	A_6	B_0	B_2	B_4	B_6
2	$-\frac{16}{19}$ $-{\cdot}8421$	$-\frac{22}{19}$ $-1{\cdot}1579$			$+\frac{16}{19}$ $+{\cdot}8421$			
4	$-\frac{32}{17}$ $-1{\cdot}8824$	$+\frac{28}{51}$ $+{\cdot}5490$	$+\frac{48}{17}$ $+2{\cdot}8235$		$+\frac{32}{17}$ $+1{\cdot}8824$	$-\frac{128}{51}$ $-2{\cdot}5098$		
6	$-\frac{32}{11}$ $-2{\cdot}9091$	$+18$ $+18{\cdot}0000$	$+\frac{184}{11}$ $+16{\cdot}7273$	$-\frac{272}{55}$ $-4{\cdot}9455$	$+\frac{32}{11}$ $+2{\cdot}9091$	$-\frac{1152}{55}$ $-20{\cdot}9455$	$+\frac{256}{55}$ $+4{\cdot}6545$	
8	$-\frac{640}{163}$ $-3{\cdot}9264$	$+\frac{10328}{163}$ $+63{\cdot}3620$	$-\frac{2112}{163}$ $-12{\cdot}9571$	$-\frac{12160}{163}$ $-74{\cdot}6012$	$+\frac{640}{163}$ $+3{\cdot}9264$	$-\frac{76800}{1141}$ $-67{\cdot}3094$	$+\frac{92160}{1141}$ $+80{\cdot}7713$	$-\frac{8192}{1141}$ $-7{\cdot}1797$
10	$-\frac{400}{81}$ $-4{\cdot}9383$	$+\frac{36130}{243}$ $+148{\cdot}6831$	$-\frac{69200}{243}$ $-284{\cdot}7737$	$-\frac{56000}{243}$ $-230{\cdot}4527$	$+\frac{400}{81}$ $+4{\cdot}9383$	$-\frac{112000}{729}$ $-153{\cdot}6352$	$+\frac{320000}{729}$ $+438{\cdot}9561$	$-\frac{51200}{243}$ $-210{\cdot}6996$
12	$-\frac{672}{113}$ $-5{\cdot}9469$	$+\frac{32316}{113}$ $+285{\cdot}9823$	$-\frac{138000}{113}$ $-1221{\cdot}2389$	$+\frac{24000}{113}$ $+212{\cdot}3894$	$+\frac{672}{113}$ $+5{\cdot}9469$	$-\frac{362880}{1243}$ $-291{\cdot}9389$	$+\frac{1881600}{1243}$ $+1513{\cdot}7570$	$-\frac{2150400}{1243}$ $-1730{\cdot}0080$

TABLE II.—The coefficients for expressing the stress R.

i	C_0	C_2	C_4	C_6	D_0	D_2	D_4	D_6
2	$+\frac{26}{19}$ $+1{\cdot}3684$	$+\frac{32}{19}$ $+1{\cdot}6842$			$-\frac{32}{19}$ $-1{\cdot}6842$			
4	$+\frac{124}{51}$ $+2{\cdot}4314$	$-\frac{112}{51}$ $-2{\cdot}1961$	$-\frac{256}{51}$ $-5{\cdot}0196$		$-\frac{128}{51}$ $-2{\cdot}5098$	$+\frac{256}{51}$ $+5{\cdot}0196$		
6	$+\frac{38}{11}$ $+3{\cdot}4545$	-24 $-24{\cdot}0000$	$-\frac{208}{11}$ $-18{\cdot}9091$	$+\frac{512}{55}$ $+9{\cdot}3091$	$-\frac{192}{55}$ $-3{\cdot}4909$	$+\frac{1536}{55}$ $+27{\cdot}9273$	$-\frac{512}{55}$ $-9{\cdot}3091$	
8	$+\frac{728}{163}$ $+4{\cdot}4663$	$-\frac{12352}{163}$ $-75{\cdot}7791$	$+\frac{4224}{163}$ $+25{\cdot}9141$	$+\frac{76288}{815}$ $+93{\cdot}6049$	$-\frac{5120}{1141}$ $-4{\cdot}4873$	$+\frac{92160}{1141}$ $+80{\cdot}7713$	$-\frac{122880}{1141}$ $-107{\cdot}6950$	$+\frac{16384}{1141}$ $+14{\cdot}3593$
10	$+\frac{1330}{243}$ $+5{\cdot}4733$	$-\frac{41200}{243}$ $-169{\cdot}5473$	$+\frac{84800}{243}$ $+348{\cdot}9712$	$+\frac{60160}{243}$ $+247{\cdot}5720$	$-\frac{4000}{729}$ $-5{\cdot}4870$	$+\frac{128000}{729}$ $+175{\cdot}5830$	$-\frac{384000}{729}$ $-526{\cdot}7490$	$+\frac{204800}{729}$ $+280{\cdot}9328$
12	$+\frac{732}{113}$ $+6{\cdot}4779$	$-\frac{35856}{113}$ $-317{\cdot}3097$	$+\frac{158400}{113}$ $+1401{\cdot}7700$	$-\frac{38400}{113}$ $-339{\cdot}8230$	$-\frac{8064}{1243}$ $-6{\cdot}4875$	$+\frac{403200}{1243}$ $+324{\cdot}3765$	$-\frac{2150400}{1243}$ $-1730{\cdot}0080$	$+\frac{2580480}{1243}$ $+2076{\cdot}0097$

TABLE III.—The coefficients for expressing the stress T.

i	E_0	E_2	E_4	E_6	F_0	F_2	F_4	F_6
2	$+\frac{6}{19}$ $+\cdot3158$							
4	$+\frac{260}{51}$ $+5\cdot0980$	$+\frac{80}{17}$ $+4\cdot7059$			$-\frac{256}{51}$ $-5\cdot0196$			
6	$+14$ $+14\cdot0000$	$-\frac{56}{11}$ $-5\cdot0909$	$-\frac{1008}{55}$ $-18\cdot3273$		$-\frac{768}{55}$ $-13\cdot9636$	$+\frac{1024}{55}$ $+18\cdot6182$		
8	$+\frac{4392}{163}$ $+26\cdot9448$	$-\frac{13248}{163}$ $-81\cdot2761$	$-\frac{10368}{163}$ $-63\cdot6074$	$+\frac{244224}{5705}$ $+42\cdot8088$	$-\frac{30720}{1141}$ $-26\cdot9238$	$+\frac{122880}{1141}$ $+107\cdot6950$	$-\frac{49152}{1141}$ $-43\cdot0780$	
10	$+\frac{10670}{243}$ $+43\cdot9095$	$-\frac{74800}{243}$ $-307\cdot8189$	$+\frac{17600}{243}$ $+72\cdot4280$	$+\frac{577280}{1701}$ $+339\cdot3769$	$-\frac{32000}{729}$ $-43\cdot8958$	$+\frac{256000}{729}$ $+351\cdot1660$	$-\frac{102400}{243}$ $-421\cdot3992$	$+\frac{409600}{5103}$ $+80\cdot2664$
12	$+\frac{7332}{113}$ $+64\cdot8850$	$-\frac{90480}{113}$ $-800\cdot7089$	$+\frac{137280}{113}$ $+1214\cdot8673$	$+\frac{99840}{113}$ $+883\cdot5398$	$-\frac{80640}{1243}$ $-64\cdot8753$	$+\frac{1075200}{1243}$ $+865\cdot0040$	$-\frac{2580480}{1243}$ $-2076\cdot0097$	$+\frac{1474560}{1243}$ $+1186\cdot2912$

If W be a 2nd, 4th, or 6th harmonic these tables give the complete expressions for P, R, and T; if W be an 8th harmonic the only further coefficients required are A_8 and C_8.

For the cases of the 10th and 12th harmonics the values in the tables are sufficient to give the stresses approximately over a wide equatorial belt, because the series for P, R, T proceed by powers of the tangent of the latitude, and the omitted terms involve high powers of that tangent. It would hardly be safe however to apply the formula—at least as regards the 12th harmonic—for latitudes greater than 15°, because the coefficients are large.

§ 3. *On the direction and magnitude of the principal stresses in a strained elastic solid.*

Let P, Q, R, S, T, U specify the stresses in a homogeneously stressed and strained elastic solid. Let l, m, n be the direction cosines of a principal stress axis.

The consideration, that at the extremity of a principal axis the normal to the stress quadric is coincident with the radius vector, gives the equations

$$(\mathrm{P}-\lambda)l+\mathrm{U}m+\mathrm{T}n=0$$
$$\mathrm{U}l+(\mathrm{Q}-\lambda)m+\mathrm{S}n=0$$
$$\mathrm{T}l+\mathrm{S}m+(\mathrm{R}-\lambda)n=0$$

These equations lead to the discriminating cubic for the determination of λ, and the solution for l, m, n is then

$$\frac{l^2}{(\mathrm{Q}-\lambda)(\mathrm{R}-\lambda)-\mathrm{S}^2}=\frac{m^2}{(\mathrm{P}-\lambda)(\mathrm{R}-\lambda)-\mathrm{T}^2}=\frac{n^2}{(\mathrm{P}-\lambda)(\mathrm{Q}-\lambda)-\mathrm{U}^2}$$

In the case considered in the preceding sections S and U vanish, and the cubic reduces to the quadratic

$$(P-\lambda)(R-\lambda)-T^2=0$$

of which the solution is

$$2\lambda=P+R\pm\sqrt{(P-R)^2+4T^2}$$

m is obviously zero and l, n are determinable from

$$l^2(P-\lambda)=n^2(R-\lambda)$$

Let

$$l=\cos\vartheta, \quad n=\sin\vartheta$$

Then it is easily proved that

$$\cot 2\vartheta=\frac{P-R}{2T} \quad \ldots\ldots\ldots\ldots \quad (27)$$

This equation gives the directions of the principal stress-axes.

The two principal stresses N_1, N_3 are the two values of λ, so that

$$\left.\begin{aligned} N_1&=\tfrac{1}{2}(P+R)+\tfrac{1}{2}\sqrt{(P-R)^2+4T^2} \\ N_3&=\tfrac{1}{2}(P+R)-\tfrac{1}{2}\sqrt{(P-R)^2+4T^2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (28)$$

and the third principal stress, which we suppose intermediate in value between N_1 and N_3, is of course Q.

When an elastic solid is in a state of stress it is supposed, in all probability with justice, that the tendency of the solid to rupture at any point is to be estimated by the form of the stress quadric. At any rate the hypothesis is here adopted that the tendency to break is to be estimated by the difference between the greatest and least principal stresses. For the sake of brevity I shall refer to the difference between the greatest and least principal stresses as "the stress-difference." This quantity I shall find it convenient to indicate by Δ.

We may also look at the subject from another point of view:—It is a well-known theorem in the theory of elastic solids that the greatest shearing stress at any point is equal to a half of the stress-difference. It is difficult to conceive any mode in which an elastic solid can rupture except by shearing, and hence it appears that the greatest shearing stress is a proper measure of the tendency to break. This measure of tendency to break is exactly one-half of the stress-difference, and it is therefore a matter of indifference whether we take greatest shearing stress or stress-difference. For the sake of comparison with experimental results as to the stresses under which wires and rods of various materials will break and crush, I have found it more convenient to use stress-difference throughout; but the results may all be reduced to shearing stresses by merely halving the numbers given.

From (28) we have then

$$\Delta=\sqrt{(P-R)^2+4T^2} \quad \ldots\ldots\ldots\ldots \quad (29)$$

and the greatest shearing stress at the same point is $\frac{1}{2}\Delta$.

§ 4. *The application of previous analysis to the determination of the stresses produced by the weight of superficial inequalities.*

I have in a previous paper shown how Sir WILLIAM THOMSON'S solution for the state of internal strain of an elastic sphere subject to bodily forces, but not acted on by any surface forces, is to be adapted to the case of a spheroid (whose small inequalities are expressed as surface harmonics) of homogeneous elastic matter, endued with the power of mutual gravitation.* THOMSON'S solution is of course directly applicable for finding the state of strain due to a true external force, such as the tide-generating influence of the moon, but this forms only a part of the complete solution when the sphere has the power of gravitation. He introduced the effects of gravitation synthetically, but for my own purposes I prefer the analytical method pursued in my paper above referred to.

Suppose that $r=a+\sigma_i$ be the equation to an harmonic spheroid of the i^{th} order, forming inequalities on the surface of the sphere, whose density is w.

Then the causes producing a state of stress and strain in the mean sphere of radius a are, first a normal traction per unit area of the surface of the sphere equal to $-gw\sigma_i$, when g is the value of gravity, and secondly the attraction of the inequalities σ_i, acting throughout the whole sphere.

The first of these causes (viz.: the weight of the mountains or continents) is shown in my paper to produce the same state of strain as would be produced in the sphere, now free from surface action, by bodily forces corresponding with a potential $-gw(r/a)^i\sigma_i$.

As regards the second of these causes (viz.: the attraction of the mountains or continents), the potential of the layer of matter σ_i on any internal point, estimated per unit volume, is $3gw(r/a)^i\sigma_i/(2i+1)$.†

Then adding these two potentials together, we see that the surface inequalities σ_i produce the same state of strain as would be caused by the bodily forces due to a potential $-2(i-1)gw(r/a)^i\sigma_i/(2i+1)$, and the surface of the sphere is now subjected to no forces.

This expression is a solid harmonic of the i^{th} degree, and therefore the analytical

* "On the Bodily Tides of Viscous and Semi-elastic Spheroids, &c.," Phil. Trans., Part I., 1879, p. 1 (see § 2). This paper treats of the state of flow of a viscous sphere, but the problem is exactly the same as that concerning elasticity here considered. It is easy to see that if a viscous sphere be deformed into the shape of a zonal harmonic, the flow of the fluid must be meridional, and from this we may conclude that in the elastic sphere the plane of greatest and least principal stresses is also meridional. This has been already assumed to be the case in the present paper.

† If we could suppose a sphere to have homogeneous elasticity but heterogeneous density, this manner of building up the effective disturbing potential would have to be somewhat modified. Such an hypothesis is somewhat absurd, and I shall regard the sphere as homogeneous. In application to the case of the earth I shall however pay attention to the smaller density of the superficial layers by halving the height of the actual continents and the depth of the actual seas.

results of the preceding sections are directly available for finding the state of stress due to continents and mountain chains.

We must in fact put

$$W_i = -\frac{2(i-1)}{2i+1} gw\left(\frac{r}{a}\right)^i \sigma_i$$

Now in the previous developments the factors involving g, w, &c., have been omitted and W_i has been put equal to a zonal harmonic which had the value unity at the equator.

If we write

$$\text{s}_i = \sin^i\theta - \frac{i^2}{2!}\sin^{i-2}\theta\cos^2\theta + \&\text{c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where θ is the colatitude, and put h as the height above the mean sphere of the elevation at the equator, then $h\text{s}_i = \sigma_i$ and

$$W_i = -\frac{2(i-1)}{2i+1}\frac{gwh}{a^i}(r^i\text{s}_i) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

W_i was in (12) put equal to $r^i\text{s}_i$.

Thus in order to apply the preceding results to finding the stresses caused in a sphere, possessing the power of gravitation, by the weight and attraction of surface inequalities expressed by

$$r = a + h\text{s}_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

we must multiply the preceding results for P, R, T, Q by

$$-\frac{2(i-1)}{2i+1}\frac{gwh}{a^i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

§ 5. *The state of stress due to ellipticity of figure or to tide-generating forces.*

When the effective disturbing potential W_i is a solid harmonic of the second degree the solution found above will give the stresses caused by oblateness or prolateness of the spheroid. It will of course also serve for the case of a rotating spheroid with more or less oblateness than is appropriate for the equilibrium figure.

When an elastic sphere is under the action of tide-generating forces the disturbing potential is a solid harmonic of the second degree, and therefore the present solution will apply to this case also.

The formula for the stress-difference admits of reduction to a simple form when $i=2$.

On substituting colatitude θ for latitude l, (22) gives

$$\text{P}-\text{R} = r^2\sin^2\theta[(A_0-C_0)+(A_2-C_2)\cot^2\theta] + a^2(B_0-D_0)$$
$$\text{T} = r^2\sin\theta\cos\theta(E_0)$$

Then substituting for A, B, C, D, E their values from Tables I., II., III., and making some simple reductions, we find

$$\left.\begin{aligned} \mathrm{P}-\mathrm{R}&=\tfrac{6}{19}[8(a^2-r^2)-r^2\cos 2\theta] \\ 2\mathrm{T}&=\tfrac{6}{19}r^2\sin 2\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Therefore by (29) in the present case, the stress-difference

$$\Delta=\tfrac{6}{19}\sqrt{64(a^2-r^2)^2+r^4-16r^2(a^2-r^2)\cos 2\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

In order to find the actual value of Δ in any special case we shall have to multiply (35) by appropriate factors; the factors will be determined below. For the present it will be more convenient to omit the factor $\frac{6}{19}$, and to reintroduce it along with these other factors.

The formula (35) enables us to determine the distribution of stress-difference throughout the sphere in the cases to which this section applies.

The curves of equal stress-difference are given by the equation

$$64(a^2-r^2)+r^4-16r^2(a^2-r^2)\cos 2\theta=\text{a constant}$$

The stress-difference at any point on any equatorial radius (for which $\theta=\frac{1}{2}\pi$) is clearly given by $8a^2-7r^2$, and along the polar radius (for which $\theta=0$) by $8a^2-9r^2$.

From this result it is clear that the stress-difference vanishes at that point on the polar radius for which $r=\frac{2}{3}a\sqrt{2}=\cdot 9425a$; this is the only point within the whole spheroid for which it vanishes.

When $r=a$ the stress-difference is equal to a^2, from which we obtain the remarkable result that the stress-difference is constant all over the surface. When $r=0$, it is equal to $8a^2$, which is eight times the surface value.

By means of arithmetical calculation and graphical interpolation I have drawn fig. 1; it shows the curves of equal stress-difference throughout a meridional section of the spheroid. The numbers written on the curves represent the values of stress-difference when the radius of the sphere is unity and when the factors above referred to are omitted (see Plate 19, fig. 1).

The point marked 0 is that in which the stress-difference vanishes. Round this point are drawn two dotted curves along which the stress-differences are $\frac{1}{2}$ and $\frac{3}{4}$ respectively. The remainder of the curves are drawn for equidistant values of stress-difference, and are marked 1, 2, 3 . . . 8. The curve marked 1 is singular, for the whole of the surface forms one branch of it, whilst there is another branch which runs below the surface from the polar axis and then rises to the surface at the point where $\cos 2\theta=-\frac{1}{8}$, that is to say, in lat. 41° 25′. Near the centre the curves are approximately circular, and they become somewhat like ellipses as we recede from the centre.

If this figure be made to rotate about the polar axis, the several curves will of course generate the surfaces of equal stress-difference throughout the sphere.

Writing ϑ for the inclination of one of the principal axes to the equator, we have by means of the formula (27)

$$\cot 2\vartheta = \frac{\mathrm{P}-\mathrm{R}}{2\mathrm{T}} = 8\left\{\left(\frac{a}{r}\right)^2 - 1\right\} \operatorname{cosec} 2\theta - \cot 2\theta$$

It would be easy to trace out the changes of direction of the principal stress-axes throughout the sphere, but I will only now make the observation that all over the surface they are parallel to and perpendicular to the surface, and that at the centre they are polar and equatorial, the stress-quadric being of course an ellipsoid of revolution.

We have next to find the actual amount of stress-difference which arises from given ellipticity of form of the spheroid. Putting $i=2$ in (30) we have

$$\varsigma_i = \sin^2\theta - 2\cos^2\theta = 3[\tfrac{1}{3} - \cos^2\theta]$$

The equation to the spheroid is

$$r = a + h\varsigma_i$$
$$= a\left[1 + 3\frac{h}{a}(\tfrac{1}{3} - \cos^2\theta)\right] = a[1 + e(\tfrac{1}{3} - \cos^2\theta)]$$

Thus $3h/a$ is the ellipticity of the spheroid, which we may put equal to e.

Then it was shown in (33) § 4 that the results for the stresses P, Q, R, &c., are to be multiplied by $-\frac{2}{5}gwh/a^2$, and this will of course be also the factor for the stress-difference Δ.

Then substituting e for $3h/a$, and introducing the factor $\frac{6}{19}$, which has been omitted in considering the distribution of stress within the spheroid, we see that ellipticity e gives a stress-difference represented by

$$\Delta = -\frac{4e}{95}gwa\sqrt{64\left(1-\left(\frac{r}{a}\right)^2\right)^2 + \left(\frac{r}{a}\right)^4 - 16\left(\frac{r}{a}\right)^2\left(1-\left(\frac{r}{a}\right)^2\right)\cos 2\theta}$$

If we estimate the forces in gravitation units the factor g must be omitted.

The expression under the square-root sign is equal to unity at the surface, and to 8 at the centre.

Thus the stress-differences, in gravitation units of force, at the surface and at the centre are $\frac{4}{95}ewa$ and $\frac{32}{95}ewa$ respectively.

To apply this to the case of the earth, take $a=637\times10^6$ c.m., and $w=5{\cdot}66$, and we find the surface and central stress-differences to be respectively $152e$ and $1214e$ metric tonnes per square centimeter.

If these numbers be multiplied by 6·34, we get the same result expressed in tons per square inch. Thus in British units these two stress-differences are $926e$ and $7698e$.

If then the ellipticity e be $\frac{1}{1000}$th, the surface and central stress-differences will be nearly 1 ton and nearly 8 tons to the square inch respectively.

From the Table VII. in § 9 it will appear that cast brass ruptures with a stress-difference of about 8 tons to the square inch.

Thus a spheroid, made of material as strong as brass, and of the same dimensions and density as the earth, would only just support an excess or deficiency of ellipticity equal to $\frac{1}{1000}$th, above or below the equilibrium ellipticity adapted for its rotation.

The following is a second example:—If the homogeneous earth (with ellipticity $\frac{1}{232}$) were to stop rotating, the stress-difference at the centre would be 33 tons per square inch.

Now suppose the cause of internal stress to be the moon's tide-generating influence, and let $m=$ moon's mass, and $c=$ moon's distance.

Then the potential under which the earth is stressed is $-\frac{3}{2}(m/c^3)(\frac{1}{3}-\cos^2\theta)wr^2$, or according to the notation of § 4 $-\frac{1}{2}(m/c^3)wr^2 s_2$.

If we took into account the elastic yielding of the earth and the weight and attraction of the tidal protuberance, this potential would have to be diminished. To estimate the diminution we must of course know the amount of elastic yielding, but as there is no means of approximating thereto, it will be left out of account.

Then it is obvious that the factor by which Δ, as given in (35), must be multiplied in order to give the stress-difference is $\frac{1}{2}mw/c^3$. Thus the surface stress-difference is $\frac{6}{19}\cdot\frac{1}{2}(m/c^3)wa^2$ in absolute force units.

Then putting M for the earth's mass, and dividing by gravity, we have $\frac{3}{19}(ma^3/Mc^3)wa$ as the surface value of Δ in gravitation units. The central value of Δ is of course eight times as great.

With the numerical data used above, $wa=3605$ metric tonnes per square c.m., and $m/M=\frac{1}{82}$, $a/c=\frac{1}{60}$, whence the surface stress-difference is 32 grammes, and the central stress-difference 257 grammes per square centimeter.

But this conclusion is erroneous for the following reason. If we suppose the moon to revolve in the terrestrial equator, and imagine that the meridian from which longitudes are measured is the meridian in which the moon stands at the instant under consideration, then the tide-generating potential is $-\frac{3}{2}(m/c^3)r^2[\frac{1}{3}-\sin^2\theta\cos^2\phi]$; this expression may be written $\frac{3}{4}(m/c^3)r^2(\frac{1}{3}-\cos^2\theta)+\frac{3}{4}(m/c^3)r^2\sin^2\theta\cos 2\phi$. The former of these terms produces a permanent increase of the earth's ellipticity, and is confused and lost in the ellipticity due to terrestrial rotation, and can produce no stress in the earth. The second term is the true tide-generating potential, but it is a sectorial harmonic, and I have failed to treat such cases. Now the first of these terms causes ellipticity in a homogeneous earth equal to $(\frac{5}{2}a/g)(\frac{3}{4}m/c^3)$ according to the equilibrium tide-theory. This ellipticity is equal to $\cdot 1039\times 10^{-6}$, an excessively small quantity. If however this permanent ellipticity does not exist (and the above investigation in reality presumes it not to exist), then there will be a superficial stress-difference equal to $152\times\cdot1039\times10^{-6}$ metric tonnes per square centimeter, and a central stress-difference of eight times as much.

Since a metric tonne is a million grammes this surface stress-difference is 16 grammes, and the central 128 grammes per square centimeter. These stress-differences are exactly the halves of those which have been computed above. Thus the remaining stress-difference which is due to the moon's tide-generating influence is 16 grammes at the surface and 128 grammes at the centre per square centimeter.

A flaw in this reasoning is that stress-difference is a non-linear function of the stresses, and therefore the stress-difference arising from the sum of two sets of bodily stresses is not the sum of their separate stress-differences.

I conceive however that the above conclusion is not likely to be much wrong.

These stresses are very small compared with those arising from the weights of mountains and continents as computed below, nevertheless they are so considerable that we can understand the enormous rigidity which Sir WILLIAM THOMSON has shown that the earth must possess in order to resist considerable tidal deformations of its mass.

§ 6. *On the stresses due to a series of parallel mountain chains.*

Having considered the case of the second harmonic, I now pass to the other extreme and suppose the order of harmonics i to be infinitely great, whilst the radius of the sphere is also infinitely great.

The equatorial belt now becomes infinitely wide, and the surface inequalities consist of a number of parallel simple harmonic mountains and valleys.

If i be infinitely large, we have from (12)

$$W_i=\rho^i\left[1-\frac{1}{2!}\left(\frac{iz}{\rho}\right)^2+\frac{1}{4!}\left(\frac{iz}{\rho}\right)^4-\&\text{c.}\right]$$

Now let ξ be the depth below the surface of the point indicated in the sphere (now infinitely large) by x, y, z.

As the formulas given above apply to the meridional plane for which $y=0$, we have $\rho=a-\xi$.

Now let $b=a/i$, then when both i and a are infinite

$$\rho^i=a^i\left(1-\frac{\xi}{a}\right)^i=a^i\epsilon^{-\xi/b}$$

and since in the limit $\rho/i=a/i=b$,

$$W_i=a^i\epsilon^{-\xi/b}\left(1-\frac{1}{2!}\frac{z^2}{b^2}+\frac{1}{4!}\frac{z^4}{b^4}-\&\text{c.}\right)$$

$$=(a^i)\epsilon^{-\xi/b}\cos\frac{z}{b}$$

This expression for W involves the infinite factor a^i, and in order to get rid of it we must now consider the factor by which it is to be multiplied, in introducing the height of the mountains and gravity.

This factor is computed in § 4; it is there shown that if $r=a+h\mathfrak{s}_i$ be a harmonic spheroid, the factor is $-2(i-1)gwh/(2i+1)a^i$.

Now if the harmonic i be of an infinitely high order, $\mathfrak{s}_i$ becomes simply $\cos z/b$, and the equation to the surface is

$$\xi=-h\cos\frac{z}{b}$$

ξ being measured downwards. Thus the harmonic spheroid $h\mathfrak{s}_i$ now represents a series of parallel harmonic mountains and valleys of height and depth h, and wave-length $2\pi b$.

The factor becomes $-gwh/a^i$, when i is infinite.

Thus the effective disturbing potential W, which is competent to produce the same state of stress and strain as the weight of the mountains and valleys, is given by

$$W=-gwh\epsilon^{-\xi/b}\cos\frac{z}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Now revert to the expressions (11) for the stresses.

When i is infinite $I=2i^2$, and they become, on changing x into $(a-\xi)$

$$\text{P}=\frac{1}{2i}(a^2-r^2)\frac{d^2W}{d\xi^2}+\frac{6(a-\xi)}{2i^2}\frac{dW}{d\xi}+\frac{3}{2i}W_i$$

$$\text{T}=-\frac{1}{2i}(a^2-r^2)\frac{d^2W}{d\xi dz}-\frac{3}{2i^2}\left((a-\xi)\frac{dW}{dz}-z\frac{dW}{d\xi}\right)$$

Now as shown above $a^2-r^2=2a\xi$, and $a/i=b$ in the limit; making these substitutions, and dropping the terms which become infinitely small when i is infinite, we have

$$\left.\begin{aligned} &\text{P}=\xi b\frac{d^2W}{d\xi^2},\ \text{T}=-\xi b\frac{d^2W}{d\xi dz} \\ &\text{R}=\xi b\frac{d^2W}{dz^2},\ \text{Q}=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and by a similar process

Then from (36) and (37) we have

$$\left.\begin{aligned} \text{P}&=-gwh\frac{\xi}{b}\epsilon^{-\xi/b}\cos\frac{z}{b} \\ \text{R}&=\ gwh\frac{\xi}{b}\epsilon^{-\xi/b}\cos\frac{z}{b} \\ \text{T}&=\ gwh\frac{\xi}{b}\epsilon^{-\xi/b}\sin\frac{z}{b}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Since the stress-difference

$$\Delta=\sqrt{(\text{P}-\text{R})^2+4\text{T}^2}$$

we have

$$\Delta=2gwh\frac{\xi}{b}\epsilon^{-\xi/b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The directions of the stress-axes are given by

$$\cot 2\vartheta = \frac{P-R}{2T} = \cot \frac{z}{b}$$

so that

$$\vartheta = \frac{1}{2}\frac{z}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Equation (39) gives the stress-difference at a depth ξ below the mean surface, and is very remarkable as showing that the stress-difference depends on depth below the mean horizontal surface and not at all on the position of the point considered with reference to the ridges and furrows.

Equation (40) shows that if we travel along uniformly horizontally through the solid perpendicular to the ridges, the stress-axes revolve with a uniform angular velocity.

They are vertical and horizontal when we are under a ridge, and they have turned through a right angle and are again vertical and horizontal by the time we have arrived under a furrow.

Since the function $x\epsilon^{-x}$ is a maximum when $x=1$, the stress-difference Δ is a maximum when $\xi=b$,—that is to say, at a depth equal to $1/2\pi$ of the wave-length—and is then equal to $2gwh\epsilon^{-1}$ or in gravitation units of force to $\cdot 736\, wh$. It is interesting to notice that the value of this maximum depends only on the height and density of the mountains, and does not involve the distance from crest to crest. The depth at which this maximum is reached depends of course on the wave-length.

Plate 19, fig. 2, shows the distribution of stress-difference, the vertical ordinates represent stress-difference, and the horizontal ones depth below the surface. The numbers written on the horizontal axis are multiples of b; the distance OL on this scale is equal to 6·28, and is therefore equal to the wave-length from crest to crest, and the distance OH is the semi-wave-length from crest to furrow.

In the case of terrestrial mountains w is about 2·8, and if we suppose h to be 2000 meters, or a little over 6000 feet, we have the case of a series of lofty mountain chains —for it must be remembered that the valleys run down to 2000 meters below the mean surface, so that the mountains are some 13,000 feet above the valley-bottoms.

Then $h=2\times 10^5$, $w=2\cdot 8$, and the maximum stress-difference is

$$\cdot 736 \times 2\cdot 8 \times 2 \times 10^5 = \cdot 412 \times 10^6 \text{ grammes per square centimeter.}$$

This stress-difference is, in British measure, 2·6 tons per square inch.

If we suppose (as is not unreasonable) that it is 314 miles from crest to crest of the mountains, then the maximum stress will be reached at 50 miles below the surface.

From Table VII., § 9, it will be seen that if the materials of the earth at this depth of 50 miles had only as much tenacity as sheet lead, the mountain chains would sink down, but they would just be supported if the tenacity were equal to that of cast tin.

§ 7. *On the stresses due to the even zonal harmonic inequalities.*

Having considered the two extreme cases where i is 2, and infinity, I pass now to the intermediate ones. As the odd zonal harmonics were not required for the investigation in the following section I have only worked out in detail the even ones.

The surface of the sphere is now corrugated by a series of undulations parallel to the equator, and the altitude of the corrugations increases towards the poles. The form of the undulation in the neighbourhood of the equator is exhibited in Plate 19, fig. 3.

The stress-difference is as before given by

$$\Delta=\sqrt{(\mathrm{P}-\mathrm{R})^2+4\mathrm{T}^2}$$

To form this expression the series in (22) for R must be subtracted from the series for P. Since the C's and D's of Table II. have always the opposite signs from the A's and B's of Table I., this algebraic subtraction becomes a numerical addition of the numbers in these two tables.

The results are given in the following table.

TABLE IV.—The coefficients for expressing P—R.

i	A_0-C_0	A_2-C_2	A_4-C_4	A_6-C_6	B_0-D_0	B_2-D_2	B_4-D_4	B_6-D_6
2	−2·2105	−2·8421			+2·5263			
4	−4·3137	+2·7451	+7·8431		+4·3922	−7·5294		
6	−6·3636	+42	+35·6364	−14·2545	+6·4	−48·8727	+13·9636	
8	−8·3926	+139·1411	−38·8712	−168·2061	+8·4137	−148·0806	+188·4663	−21·5390
10	−10·4115	+318·2304	−633·7449	−478·0247	+10·4252	−329·2182	+965·7051	−491·6324
12	−12·425	+603·292	−2623·009	+552·212	+12·434	−616·315	+3243·765	−3806·018

Then we have

$$\mathrm{P}-\mathrm{R}=r^i\cos^i l[(A_0-C_0)+(A_2-C_2)\tan^2 l+\ldots]$$
$$+a^2r^{i-2}\cos^{i-2} l[(B_0-D_0)+(B_2-D_2)\tan^2 l+\ldots]$$

The materials for computing T have been already given in Table III.

The series for P—R and for 2T should now be squared and added together, but the result would be so complex that it is not worth while to proceed algebraically.

At the equator T=0, and Δ=P—R, and P—R reduces to only two terms, whatever be the order of harmonic.

By reference to (23) and (24) we see that at the equator

$$\Delta = \frac{i r^{i-2}}{2(i+1)^2+1}\left[\frac{i(i+2)(2i-1)}{i-1}a^2-(i+1)(2i+3)r^2\right]$$

or

$$\Delta = \frac{i(i+1)(2i+3)a^2 r^{i-2}}{(i+1)(2i+3)-i}\left[1-\left(\frac{r}{a}\right)^2+\frac{3}{(i^2-1)(2i+3)}\right] \quad . \quad . \quad . \quad . \quad . \quad (41)$$

This value for Δ requires of course multiplication by appropriate factors involving the height of the continents and gravity.

Even when i is no larger than 6, (41) differs but little from $ir^{i-2}(a^2-r^2)$, at least for values of r not very nearly equal to a.

Δ clearly reaches a maximum when

$$\left(\frac{r}{a}\right)^2 = \frac{i-2}{i}\left\{1+\frac{3}{(i^2-1)(2i+3)}\right\}$$

For large values of i this maximum is nearly equal to $2\{(i-2)/i\}^{\frac{1}{2}i-1}a^i$

From these formulas the following results are easily obtained.

TABLE V. (a).

$i=$	2	4	6	8	10	12
Maximum value of Δ . . .	2·526	1·118	·959	·894	·859	·836
Value of r/a when Δ is max. .	0	·714	·819	·867	·895	·913

Plate 20, fig. 4, shows graphically the law of diminution of stress-difference for the even zonal harmonics, the vertical ordinates representing stress-difference and the horizontal ones the distances from the surface or from the centre of the globe.

In order to find a numerical value of these maximum stress-differences which shall be intelligible according to ordinary mechanical ideas, I will suppose the height of each of the harmonics at the equator to be 1500 meters. On account of the small density of the superficial layers in the earth, this is very nearly the same as supposing that in the earth the maximum height of the continents above, and the maximum depth of the depressions below the mean level of the earth are each about 3000 meters. In the summary at the end I shall show that there is reason to believe that this is about the magnitude of terrestrial inequalities.

Then by (33) we have to multiply the maximum stress-differences in the above table by $2(i-1)wh/(2i+1)$, in order to obtain the stress-differences for the supposed continents in grammes or tonnes per square centimeter.

Now $w=5{\cdot}66$ and $h=1{\cdot}5\times10^5$ according to the above hypothesis as to height of continent; and the coefficient in i is of course different for each harmonic.

By performing these multiplications I find the following results.

TABLE V. (*b*).—Maximum stress-differences due to harmonic continents and seas.

Order of harmonic.	2	4	6	8	10	12
Max. stress-difference, in metric tonnes per sq. c.m. due to continents 1500 meters high	·858	·633	·626	·625	·625	·625
Ditto in British tons per sq. inch, for same continents.	5·43	4·01	3·97	3·96	3·96	3·96
Depth in British miles at which this stress is attained	Centre of earth	1146	725	532	420	347

N.B.—The continents referred to are supposed to be of the earth's mean density and are equivalent to actual continents of double the height.

Thus far we have only considered the stress-differences at the equator immediately underneath the centres of the continents, but we must now see how they differ as the latitude of the place of observation increases. In order to attain this result a good deal of computation was necessary.

For this purpose I calculated P—R and 2T for a number of points and found the square root of the sum of these squares. As the computations were laborious, and as the results given in the following table are amply sufficient for the purpose in hand, I did not think it worth while to trace the changes to a greater depth than $r=\cdot 7$. Moreover the correctness of the last significant figures given cannot be guaranteed, although I believe that it is correct in most cases.

TABLE VI.—Showing the stress-difference due to the several harmonic inequalities at various depths and in various latitudes.

i		Equator.	Lat. 6°.	Lat. 12°.	i		Equator.	Lat. 6°.	Lat. 12°.
2	$r=1.$	·316	·316	·316	8	$r=1.$	·021	·015	·000
	$r=\cdot 9$	·736	·732	·721		$r=\cdot 9$	·859	·853	·853
	$r=\cdot 8$	1·112	1·108	1·097		$r=\cdot 8$	·798	·795	·797
	$r=\cdot 7$	1·443	1·440	1·431		$r=\cdot 7$	·506	·505	·507
4	$r=1.$	·079	·074	·061	10	$r=1.$	·014	·008	·007
	$r=\cdot 9$	·727	·719	·700		$r=\cdot 9$	·857	·854	·860
	$r=\cdot 8$	1·044	1·038	1·025		$r=\cdot 8$	·631	·630	·635
	$r=\cdot 7$	1·116	1·113	1·104		$r=\cdot 7$	·307	·307	·309
6	$r=1.$	·036	·031	·016	12	$r=1.$	·010	·003	·019
	$r=\cdot 9$	·817	·810	·800		$r=\cdot 9$	·827	·824	·835
	$r=\cdot 8$	·953	·949	·945		$r=\cdot 8$	·481	·481	·486
	$r=\cdot 7$	·788	·786	·785		$r=\cdot 7$	·179	·179	·181

The numbers given in the column marked "equator" might be computed from (41);

and are those exhibited graphically in fig. 4; they are here given as a means of comparison with the numbers corresponding to latitudes 6° and 12°.

The result to be deduced from this table is that the lines of equal stress-difference are very nearly parallel with the surface, and that it is for all practical purposes sufficient to know the stress-difference immediately under the centre of the continents.

We have already seen in § 6 that for harmonics of infinitely high orders the lines of equal stress-difference are rigorously parallel with the mean surface.

§ 8. *On the stresses due to the weight of an equatorial continent.*

The actual continents and seas on the earth's surface have not got quite the regular wavy character of the elevations and depressions which have been treated hitherto. The subject of the present section possesses therefore a peculiar interest for the purpose of application to the earth. Had I however foreseen, at the beginning, the direction which the results of this whole investigation would take, it is probable that I might not have carried out the long computations which were required for discussing the case of an isolated continent. But now that that end has been reached, it seems worth while to place the results on record.

The function exp.$[-\cos^2\theta/\sin^2\alpha]$ (where θ is colatitude) obviously represents an equatorial belt of elevation, and I therefore chose it as the form of the required equatorial continent. This function has to be expanded in a series of zonal harmonics in order to apply the analytical solutions for the stresses produced by the weight of the continent.

It is obvious by inspection that the odd zonal harmonics can take no part in the representation of the function, and it was on this account that I have above only worked out the cases of the even zonal harmonics.

The multiplication of this function by the successive LEGENDRE'S functions, and integration over the surface of the sphere, are operations algebraically tedious, and wholly uninteresting, and I therefore simply give the results.

I find then that if $\alpha=10°$, and

$$\mathrm{s}_i=\sin^i\theta-\frac{i^2}{2!}\sin^{i-2}\theta\cos^2\theta+\frac{i^2(i-2)^2}{4!}\sin^{i-4}\theta\cos^4\theta-\&c.$$

Then

$$\left.\begin{array}{l} 2\epsilon^{-\cos^2\theta/\sin^2\alpha}-\beta_0=\beta_2\mathrm{s}_2+\beta_4\mathrm{s}_4+\beta_6\mathrm{s}_6+\beta_8\mathrm{s}_8+\beta_{10}\mathrm{s}_{10}+\beta_{12}\mathrm{s}_{12}+\ldots \\ \text{where} \\ \beta_0=\cdot3078,\ \beta_2=\cdot3673,\ \beta_4=\cdot3339,\ \beta_6=\cdot2829,\ \beta_8=\cdot2252,\ \beta_{10}=\cdot1688, \\ \quad\beta_{12}=\cdot1193 \end{array}\right\} \quad . \quad (42)$$

β_0 is put on the left-hand side in order that the mean value of the function may be zero. The β's obviously decrease very slowly, and as I stop with the 12th harmonic, the representation of the function is very imperfect.

Plate 20, fig. 5, illustrates the results of the representation, the portion of a circle marked "mean level of earth" represents a meridional section of the earth; the dotted curve marked "inequality to be represented" shows the true value of the function $2\exp.[-\cos^2\theta/\sin^2\alpha]-\beta_0$; the curve marked "representation" shows the right-hand side of (42) stopping with the 12th harmonic; the second curve is the same without the 2nd harmonic constituent $\beta_2 \mathfrak{s}_2$, and it is introduced for the reasons explained in the discussion and summary at the end.

The equatorial value of the exponential function is 1·792, that of the "representation" is 1·497, and that of "the representation without the 2nd harmonic" is 1·130.

The polar value of the exponential is −·3078, that of the "representation" is −·0830, and that of "the representation without 2nd harmonic" is +·6516. This latter function thus gives us an equatorial continent and a pair of polar continents of less height.

The extreme difference of height in the "representation" between the elevation at the equator and the depression at the pole is $(1·497+·083)h$ or $1·58h$. I do not exactly know the extreme difference in the case where the 2nd harmonic is omitted, because I have not traced the inequality throughout, but it is probably about 1·3 or $1·4h$.

Now let Δ_i be the numerical value, as computed for § 7, of the stress-difference due to the harmonic spheroid $\mathfrak{s}_i$. Then it has been shown above that the stress-difference due to the spheroid whose equation is $r=a+h\mathfrak{s}_i$ is $-2(i-1)gwh\Delta_i/(2i+1)$.

Now stress-difference is a non-linear function of the component stresses P, R, T, and therefore the stress-difference due to a compound harmonic spheroid is not in general the sum of the stress-differences due to the constituent harmonic spheroids. At any point, however, where the principal stress-axes are all coincident in direction and where all the greater stress-axes coincide, and not a greater with a less, and where $T=0$, the stress-difference is linear and is the sum of the constituent stress-differences. This is the case at the equator for the present equatorial continent.

Hence, if Δ be the stress-difference at the equator due to the spheroid,

$$r=a+h(\beta_2\mathfrak{s}_2+\beta_4\mathfrak{s}_4+\ldots+\beta_{12}\mathfrak{s}_{12})$$

We have

$$\Delta=-gwh[\tfrac{2}{5}\beta_2\Delta_2+\tfrac{6}{9}\beta_4\Delta_4+\tfrac{10}{13}\beta_6\Delta_6+\tfrac{14}{17}\beta_8\Delta_8+\tfrac{18}{21}\beta_{10}\Delta_{10}+\tfrac{22}{25}\beta_{12}\Delta_{12}] \quad . \quad . \quad . \quad (43)$$

In this formula the Δ_i's are the numbers which were computed for drawing Plate 20, fig. 4, from the formula (41), namely

$$\Delta_i=-\frac{i(i+1)(2i+3)}{2(i+1)^2+1}\left(\frac{r}{a}\right)^{i-2}\left[1-\left(\frac{r}{a}\right)^2+\frac{3}{(i^2-1)(2i+3)}\right]$$

By using these computations I have drawn Plate 20, fig. 6. The vertical ordinates are $-\Delta\div gwh$, and the horizontal are the distances from surface or centre of the sphere.

The maxima in the two curves are merely found graphically, and the distances where the maxima are reached (viz.: 660 and 590 miles from the surface) are written down on the supposition that the radius of the sphere is 4000 miles.

In the discussion in the second part of this paper, I have endeavoured to make an estimate of the proper elevation to attribute to these isolated continents; so as to make the case, as nearly as may be, analogous to the earth.

Although it appears impossible to make an accurate estimate, I conclude that it will not be excessive if we assume that the greatest difference of height, between the highest point in the equatorial elevation and the approximately spherical remainder of the globe, is 2000 meters.

Accordingly for the representation I put $1{\cdot}58h=2000$, and for the second curve $1{\cdot}4h=2000$; these give $h=1{\cdot}27\times10^5$ c.m. and $h=1{\cdot}4\times10^5$ c.m. respectively.

Taking $w=5{\cdot}66$, then for the representation we have $wh={\cdot}72$ tonnes per square centimeter, and for the other curve $wh={\cdot}79$ of the same units. The maximum stress-differences are ${\cdot}91wh$ and ${\cdot}76wh$ respectively.

Therefore for the equatorial table-land (called above the representation) we have a maximum stress-difference of ·66 metric tonnes per square c.m. or 4·1 British tons per square inch; and for the equatorial table-land balanced by a pair of polar continents (2nd harmonic omitted) we have a maximum stress-difference of ·60 tonne per square c.m. or 3·8 tons per square inch.

I therefore conclude that our great continental plateaus produce a stress-difference of about 4 tons per square inch at a depth of 600 or 700 miles from the earth's surface.

§ 9. *On the strength of various substances.*

In order to have a proper comprehension of the strength which the earth's mass must possess in order to resist the tendency to rupture, produced by the unequal distribution of weights on the surface, it is necessary to consider the results of experiments.

RANKINE* gives a large number of results obtained by various experimenters, and Sir WILLIAM THOMSON also gives similar tables in his article on 'Elasticity.'†

Amongst other constants Sir WILLIAM gives YOUNG'S modulus and the greatest elastic extension. If the materials of a wire remain perfectly elastic when the wire is just on the point of breaking under tension, then the product of YOUNG'S modulus into the greatest elastic extension should be equal to what is called the tenacity, which is defined as the breaking tension per square centimeter of the area of the wire.

* 'Useful Rules and Tables:' GRIFFIN, London, 1873, p. 191, *et seq.*

† 'Elasticity:' BLACK, Edinburgh, 1878. This is the article from the 'Encyclopedia Britannica,' but it is also published as a separate work.

If however a permanent set begins to take place before the wire breaks, this product should be less than the tenacity. I do not see how it can ever be greater, unless there be a marked departure from HOOK's law "ut tensio sic vis;" or different sets of experiments with the same class of material might make it seem greater. In some of the results given by Sir WILLIAM THOMSON the product of modulus and elastic extension is however greater than tenacity.

Ordinary experience would lead one to suppose that such materials as lead and copper would undergo a considerable stress beyond the limits of perfect elasticity, before breaking. It is surprising therefore to see how nearly identical this product is to the tenacity—indeed in the case of lead absolutely identical, as may be seen in the table below.

With regard to the earth we require to know what is the limiting stress-difference under which a material takes permanent set or begins to flow, rather than the stress-difference under which it breaks; for if the materials of the earth were to begin to flow, the continents would sink down and the sea bottoms rise up.

It will be seen from the definition of tenacity given above that it is the rupturing stress-difference for tensional stresses. There is no word specially applied to rupturing stress-difference under pressure.

I am inclined to think that for the purposes of this investigation these tables in most cases rate the strength of materials somewhat too highly; for it seems probable that a permanent set would be taken, if a material were subjected for a long time to a stress-difference, which is a considerable fraction of the limiting value. We are likely to know more on this point in some years time when the wires hung by Sir WILLIAM THOMSON in the tower of Glasgow University have been subjected to several years of tension. However this may be I give the results of some of the experiments as collected and quoted by Sir WILLIAM THOMSON and the late Professor RANKINE. The first table of tenacity, except the results denoted by the letter R, are taken from Sir WILLIAM THOMSON. The second table of crushing stress-difference is taken entirely from RANKINE. The multiplications and reductions to different units I have done myself.

TABLE VII.—Limiting stress-difference.

Produced by tension.				Produced by crushing.		
Material.	Breaking stress-difference in metric tonnes per square centimeter.	Stress-difference at which permanent set begins in—		Material.	Breaking stress-difference in—	
		Metric tonnes per square centimeter.	British tons per square inch.		Metric tonnes per square centimeter.	British tons per square inch.
Sheet lead. . . .	·23	·23	1·46	Strong red brick .	·077	·49
Cast tin	·416	·417	2·64	Strong sandstone .	·39	2·45
,, ...	·325 (R)	. .	2·06 (R)	(F) Strong limestone	·60	3·80
Wood (ash) . . .	1·20	1·20	7·61	Marble	·39	2·45
Cast brass. . . .	1·27	1·27	8·05	Granite	·39 to ·77	2·45 to 4·91
,, iron	·94 to 2·04	1·14 to 1·87	7·23 to 11·86	(F) Granite (Mount Sorrel)	·905	5·74
Drawn copper . .	4·10	4·00	25·36	(F) Grauwacke. .	1·19	7·54
English steel pianoforte wire . . .	23·62	23·56	149·6	Ash (along the grain)	·63	4·02
(R) Brick, cement .	·020 to ·021	. .	·125 to ·134	Cast brass . . .	·73	4·60
(R) Glass	·66	. .	4·20	Wrought iron . .	2·52 to 2·84	16 to 18
(R) Slate	·68 to ·90	. .	4·3 to 5·7			

NOTE.—The second and third columns give the product of YOUNG's modulus into greatest elastic extension, and this should give the stress-difference when permanent set begins. RANKINE does not give the data for this quantity, but the breaking stress-difference is given in both metric and British units, the latter being in the third column.

In the second half of the table the results marked F are from Sir WILLIAM FAIRBAIRN's experiments.

The only cases in these two tables in which we have the opportunity of comparing the strength for resisting the stress-difference, when produced in the two manners, is for the materials cast brass and ash; in both cases we see that the substance is considerably weaker for crushing than for tension.

I should be inclined to suppose that the crushing strength is more nearly the datum we require for the case of the stresses in the earth.

In the first half of the table we probably see the effect of permanent set in the cases of copper and pianoforte wire (compare 4·00 with 4·10, and 23·56 with 23·62), but it is surprising that the contrast between the two columns is not more marked.

§ 10. *On the case when the elastic solid is compressible.*

It appears desirable to know how far the results of the preceding investigation may differ, if the elastic solid be compressible. According to the views of Dr. RITTER, referred to in the summary, this must be largely the case.

As I did not examine this point until after the above was completed, it seemed preferable to make a fresh beginning, rather than to modify the whole investigation.

The process is however so closely analogous, that it presents no difficulty, and may be dismissed shortly. I shall accordingly follow closely the processes of §§ 1 and 2.

If a solid be very compressible it takes a comparatively small hydrostatic pressure to produce a given amount of compression; that is to say, although the compressibility is large, the modulus of compressibility is small compared with that of rigidity. The modulus of compressibility I shall call the incompressibility. In the preceding investigation the converse was the case, for the incompressibility was infinite compared with the rigidity. By the definitions of ω and υ in § 1, the incompressibility

$$k=\omega-\tfrac{1}{3}\upsilon.$$

It will be found convenient to use k and ω as the two moduli, which define the nature of the elastic solid.

In the denominators in E_i, F_i, G_i of (1), the expression $(2(i+1)^2+1)\omega-(2i+1)\upsilon$ occurs, this I shall call K, in analogy with I.

Then

$$K=2i(i-1)\omega+3(2i+1)k$$

If we develop the last differential coefficient in the expression (1) for α, we find

$$2\upsilon K\alpha=3k\left(\frac{i}{i-1}a^2-r^2\right)\frac{dW}{dx}+i\omega\left[(a^2-r^2)\frac{dW}{dx}+2xW\right] \quad . \quad . \quad . \quad . \quad (44)$$

Also

$$K\delta=iW \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and

$$\left.\begin{aligned} \mathrm{P}&=(3k-2\omega)\delta+2\upsilon\frac{d\alpha}{dx} \\ \mathrm{T}&=\upsilon\left(\frac{d\alpha}{dz}+\frac{d\gamma}{dx}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Differentiating (44) with regard to x, and substituting in (46) we find

$$K\mathrm{P}=3k\left[\left(\frac{i}{i-1}a^2-r^2\right)\frac{d^2W}{dx^2}-2x\frac{dW}{dx}+iW\right]+i\omega(a^2-r^2)\frac{d^2W}{dx^2} \quad . \quad . \quad . \quad (47)$$

Again, differentiating (44) with regard to z, writing down $d\gamma/dx$ by symmetry, and adding the two together, we have from (46)

$$K\mathrm{T}=3k\left[\left(\frac{i}{i-1}a^2-r^2\right)\frac{d^2W}{dxdz}-z\frac{dW}{dx}-x\frac{dW}{dz}\right]+i\omega(a^2-r^2)\frac{d^2W}{dxdz} \quad . \quad . \quad . \quad (48)$$

From (47) and (48) the other stresses may be written down by cyclic changes of letters.

Now let us suppose that the incompressibility is small compared with the rigidity.

Then ω becomes nearly equal to $\frac{1}{3}v$, and k is small compared with v and ω. Also K becomes nearly equal to $2i(i-1)\omega$. Hence in this case

$$\left.\begin{aligned}\text{P}&=\frac{1}{2(i-1)}(a^2-r^2)\frac{d^2W}{dx^2}\\ \text{T}&=\frac{1}{2(i-1)}(a^2-r^2)\frac{d^2W}{dxdz}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Now if we suppose W to be a zonal harmonic of degree i, and only consider the state of stress at the equator, immediately underneath the centre of the elevation, then T is zero and by (15)

$$\text{P}=\frac{i(i-1)}{2(i-1)}(a^2-r^2)r^{i-2}$$

$$\text{R}=-\frac{i^2}{2(i-1)}(a^2-r^2)r^{i-2}$$

And

$$\Delta=\text{P}-\text{R}=\frac{i(2i-1)}{2(i-1)}(a^2-r^2)r^{i-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

If i be large $\Delta=i(a^2-r^2)r^{i-2}$.

In § 7 we have found this identical result, under the like conditions, when the solid is incompressible.

Now take the case of the 2nd harmonic, so that

$$W=r^2(\tfrac{1}{3}-\cos^2\theta)=\tfrac{1}{3}(x^2+y^2-2z^2)$$

And we have

$$\begin{aligned}\text{P}&=\ \tfrac{1}{3}(a^2-r^2)\\ \text{Q}&=\ \tfrac{1}{3}(a^2-r^2)\\ \text{R}&=-\tfrac{2}{3}(a^2-r^2)\\ \text{T}&=\ 0\end{aligned}$$

Thus throughout the spheroid, the principal stress-axes are parallel and perpendicular to the polar axis; also

$$\Delta=(a^2-r^2)$$

Hence the central stress-difference is a^2. If the solid be incompressible it was found to be $8a^2$. Hence infinite compressibility largely relieves the stress-difference due to ellipticity of figure. Next take the case when the harmonic is of the 4th order.

Then at the equator we have by (50)

$$\Delta=\tfrac{4\cdot 7}{6}r^2(a^2-r^2)=\tfrac{14}{3}r^2(a^2-r^2)$$

The maximum is reached, when $r^2=\frac{1}{2}a^2$ or $r=\cdot 707a$, and is equal to $\frac{7}{6}=1\cdot 167$. Comparing this with the Table V. (a) § 7, we see that infinite compressibility makes very little difference, for ·707 differs little from ·714 and 1·167 from 1·118.

It will now be shown that whatever may be the compressibility of the solid, we get identically the same solution in the case when the harmonic is of an infinitely high order. This is the problem of the harmonic mountains and valleys corrugating a mean level surface, which was considered in § 6. The same notation will be adopted here.

Both i and a are infinite, x becomes $a-\xi$, and $ia^2/(i-1)-r^2=a^2-r^2=2a\xi$. If the substitutions here suggested be made in (47) and (48), it will be found that the terms with ω as a factor are multiplied by ai (two infinites), whilst none of the terms with k as a factor involve more than one infinite. Hence the latter terms are negligeable compared with the former.

Again i being infinite, $K=2i^2\omega$. Thus if i and a be infinite (47) and (48) reduce to

$$\mathrm{P}=\frac{1}{2i^2}(i2a\xi)\frac{d^2W}{d\xi^2},\quad \mathrm{R}=\frac{1}{2i^2}(i2a\xi)\frac{d^2W}{dz^2},\quad \mathrm{T}=-\frac{1}{2i^2}(i2a\xi)\frac{d^2W}{d\xi dz}$$

but $a/i=b$, therefore

$$\mathrm{P}=b\xi\frac{d^2W}{d\xi^2},\quad \mathrm{R}=b\xi\frac{d^2W}{dz^2},\quad \mathrm{T}=-b\xi\frac{d^2W}{d\xi dz}$$

In (36) we found the effective potential, which produced the same state of stress as the weight of the mountains whose equation was $-\xi=h\cos z/b$; the result was

$$W=-gwh\,\epsilon^{-\xi/b}\cos\frac{z}{b}$$

Hence in the present case

$$\mathrm{P}=-gwh\frac{\xi}{b}\epsilon^{-\xi/b}\cos\frac{z}{b},\quad \mathrm{R}=gwh\frac{\xi}{b}\epsilon^{-\xi/b}\cos\frac{z}{b},\quad \mathrm{T}=gwh\frac{\xi}{b}\epsilon^{-\xi/b}\sin\frac{z}{b}$$

These results are identical with equations (38), which gave the stresses when the elastic solid was incompressible.

It may be concluded from this that, except for the case of the 2nd harmonic inequality, compressibility makes very little difference, and for the higher harmonics it makes no difference at all.

II.

SUMMARY AND DISCUSSION.

The existence of dry land proves that the earth's surface is not a figure of equilibrium appropriate for the diurnal rotation.

Hence the interior of the earth must be in a state of stress, and as the land does not sink in, nor the sea-bed rise up, the materials of which the earth is made must be strong enough to bear this stress.

We are thus led to inquire how the stresses are distributed in the earth's mass, and what are their magnitudes. These points cannot be discussed without an hypothesis as to the interior constitution of the earth.

In this paper I have solved a problem of the kind indicated for the case of a homogeneous incompressible elastic sphere, and have applied the results to the case of the earth.

It may of course be urged that the earth is not such as this treatment postulates.

The view which was formerly generally held was that the earth consists of a solid crust floating on a molten nucleus. It has also been lately maintained by Dr. AUGUST RITTER in a series of interesting papers that the interior of the earth is gaseous.* A third opinion, contended for by Sir WILLIAM THOMSON, and of which I am myself an adherent, is that the earth is throughout a solid of great rigidity; he explains the flow of lava from volcanoes either by the existence of liquid vesicles in the interior, or by the melting of solid matter, existing at high temperature and pressure, at points where diminution of pressure occurs.

There is another consideration, which is consistent with Sir WILLIAM THOMSON'S view, and which was pointed out to me by Professor STOKES. It may be that underneath each continent there is a region of deficient density; then underneath this region there would be no excess of pressure.

For the present investigation it is to some extent a matter of indifference as to which of these views is correct, for if it is only the crust of the earth which possesses rigidity, or if Professor STOKES'S suggestion of the regions of deficient density be correct, then the stresses in the crust or in the parts near the surface must be greater than those here computed—enormously greater if the crust be thin,† or if the region of deficient density be of no great thickness.

With regard to the property of incompressibility which is here attributed to the elastic sphere, it appears from §10 that even if we suppose the elastic solid to be *very* highly compressible, yet the results with regard to the internal stresses are almost the same as though it were incompressible. I think the hypothesis of great incompressibility is likely to be much nearer to the truth than is that of great compressibility. I shall therefore adhere to the supposition of infinite incompressibility, bearing in mind that even great compressibility would not much affect most of the results.

I take then a homogeneous incompressible elastic sphere, and suppose it to have the

* 'Anwendung der mechanischen Wärmetheorie auf kosmologische Probleme.' CARL RÜMPLER, Hannover, 1879. This is a reprint of six papers in WIEDEMANN'S Annalen.

Dr. RITTER contends that the temperature in the interior of the planet is above the critical temperature and that of dissociation for all the constituents, so that they can only exist as gas. Data are wanting with regard to the mechanical properties of matter at, say 10,000° Fahr., and a pressure of many tons to the square inch. Is it not possible that such "gas" may have the density of mercury and the rigidity and tenacity of granite? Although such a conjectural "gaseous" solid might possess high rigidity, it almost certainly would have great compressibility: but it is proved in § 10 that the compressibility will make exceedingly little difference in the result of the present investigation excepting in the case of the 2nd harmonic inequality.

† The evaluation of the stresses in a crust, with fluid beneath, would be tedious, but not more difficult than the present investigation. I may perhaps undertake this at some future time.

power of gravitation and to be superficially corrugated. In consequence of mathematical difficulties the problem is here only solved for the particular class of surface inequalities called zonal harmonics, the nature of which will be explained below.

Before discussing the state of stress produced by these inequalities, it will be convenient to explain the proper mode of estimating the strength of an elastic solid under stress.

At any point in the interior of a stressed elastic solid there are three lines mutually at right angles, which are called the principal stress-axes. Inside the solid at the point in question imagine a small plane (say a square centimeter or inch) drawn perpendicular to one of the stress-axes; such a small plane will be called an inter-face.* The matter on one side of the ideal inter-face might be removed without disturbing the equilibrium of the elastic solid, provided some proper force be applied to the inter-face; in other words, the matter on one side of an inter-face exerts a force on the matter on the other side. Now a stress-axis has the property that this force is parallel to the stress-axis to which the inter-face is perpendicular. Thus along a stress-axis the internal force is either purely a traction or purely a pressure. Treating pressures as negative tractions, we may say that at any point of a stressed elastic solid, there are three mutually perpendicular directions along which the stresses are purely tractional. The traction which must be applied to an inter-face of a square centimeter in area, in order to maintain equilibrium when the matter on one side of the inter-face is removed, is called a principal stress, and is of course to be measured by grammes weight per square centimeter.

If the three stresses be equal and negative, the matter at the point in question is simply squeezed by hydrostatic pressure, and it is not likely that in a homogeneous solid any simple hydrostatic pressure, *absolutely* equal in all directions, would ever rupture the solid. The effect of the equality of the three stresses when they are positive and tractional is obscure, but at least physicists do not in general suppose that this is the cause of rupture when a solid breaks.

If the three principal stresses be unequal, one must of course be greatest and one least, and there is reason to suppose that tendency of the solid to rupture is to be measured by the difference between these principal stresses.

In one very simple case we know that this is so, for if we imagine a square bar, of which the section is a square centimeter, to be submitted to simple longitudinal tension, then two of the principal stresses are zero (namely, the stresses perpendicular to the faces of the rod), and the third is equal to the longitudinal traction. The traction under which the rod breaks is a measure of its strength, and this is equal to the difference of principal stresses.

If at the same time the rod were subjected to great hydrostatic pressure, the breaking load would be very little, if at all affected; now the hydrostatic pressure subtracts

* This term is due to Professor JAMES THOMSON.

the same quantity from all three principal stresses, but leaves the difference between the greatest and least principal stresses the same as before.

Difference of principal stresses may also be produced by crushing.

In this paper I call the difference between the greatest and least principal stresses the "stress-difference," and I say that, if calculation shows that the weight of a certain inequality on the surface of the earth will produce such and such stress-difference at such and such a place, then the matter at that place must be at least as strong as matter which will break when an equal stress-difference is produced by traction or crushing.

I shall usually estimate stress-difference by metric tonnes (a million grammes) per square centimeter, or by British tons per square inch.

In Table VII., § 9, are given the experimentally determined values of the breaking stress-difference for various substances. The table is divided into two parts, in the former of which the stress-difference was produced by tension, and in the latter by crushing. It is not necessary here to advert to the difference in meaning of the numbers given in the first column and those given in the two latter columns in the first half of the table.

The cases of wood and cast brass are the only ones where a comparison is possible between the two breaking stress-differences, as differently produced. It will be seen that the material is weaker for crushing than for tension. For the reasons given in that section, I am inclined to think that these tables rate the strength of the materials somewhat too highly for the purposes of this investigation. I conceive that the results derived from crushing are more appropriate for the present purpose than those derived from tension; and fortunately the results for various kinds of rocks seem to have been principally derived from crushing stresses.

This table will serve as a means of comparison with the numerical results derived below, so that we shall see, for example, whether or not at 500 miles from the surface the materials of the earth are as strong as granite.

We may now pass to the mathematical investigation. It appears therefrom that the distribution of stress-difference is quite independent of the absolute heights and depths of the inequalities. Although the questions of distribution and magnitude of the stresses are thus independent, it will in general be convenient to discuss them more or less simultaneously.

The problem has only been solved for the class of superficial inequalities called zonal harmonics, and their nature will now be explained.

A zonal harmonic consists of a series of undulations corrugating the surface in parallels of latitude with reference to some equator on the globe; the number of the undulations is estimated by the order of the harmonic. The harmonic of the second order is the most fundamental kind, and consists of a single undulation forming an elevation round the equator, and a pair of depressions at the poles of that equator; it may also be defined as an elliptic spheroid of revolution, and the absolute magnitude is measured by the ellipticity of the spheroid.

If the order of the harmonic be high, say 30 or 40, we have a regular series of mountain chains and intervening valleys running round the sphere in parallels of latitude.

For the sake of convenience I shall always speak as though the equator were a region of elevation, but the only effect of changing elevations into depressions, and *vice versâ*, is to diametrically reverse the directions of all the stresses.

The harmonics of the orders 2, 6, 10, &c., have depressions at the poles of the sphere; those of orders 4, 8, 12, &c., have elevations at the pole.

The harmonic of the fourth order consists of an equatorial continent and a pair of circular polar continents, with an intervening depression. That of the sixth order consists of an equatorial continent and a pair of annular continents in latitudes (about) 60° on one and the other side of the equator. The 8th harmonic brings down these new annular continents to about latitude 45°, and adds a pair of polar continents; and so on.

By a continuation of this process the transition to the mountain chains and valleys is obvious.

In § 5 the case of the 2nd harmonic is considered. As above explained the sphere is deformed into a spheroid of revolution. The investigation also applies to the case of a rotating spheroid, such as the earth, with either more or less oblateness than is appropriate for the figure of equilibrium.

The lines throughout a meridional section of the spheroid along which the stress-difference is constant are shown in Plate 19, fig. 1, and the numbers written on the curves give the relative magnitude of the stress-difference.

It is remarkable that the stress-difference is the same all over the surface. In the polar regions the stress-difference diminishes as we descend into the spheroid and then increases again; in the equatorial regions it always increases as we descend. The maximum value is at the centre, and there the stress-difference is eight times as great as at the surface.

If the elastic solid be highly compressible the stress-differences are not nearly so great as on the hypothesis of incompressibility. In all the other cases considered in this paper compressibility makes practically no difference in the results.

On evaluating the stress-difference, on the hypothesis of incompressibility, arising from given ellipticity in a spheroid of the size and density of the earth, it appears that if the excess or defect of ellipticity above or below the equilibrium value (namely $\frac{1}{232}$ for the homogeneous earth) were $\frac{1}{1000}$, then the stress-difference at the centre would be 8 tons per square inch, and accordingly, if the sphere were made of material as strong as brass (see Table VII.), it would be just on the point of rupture. Again if the homogeneous earth, with ellipticity $\frac{1}{232}$, were to stop rotating, the central stress-difference would be 33 tons per square inch, and it would rupture if made of any material excepting the finest steel.

A rough calculation* will show that if the planet Mars has ellipticity $\frac{1}{80}$ (about twice the ellipticity on the hypothesis of homogeneity) the central stress-difference must be 6 tons per square inch. It was formerly supposed that the ellipticity of the planet was even greater than $\frac{1}{80}$, and even if the latest telescopic evidence had not been adverse to such a conclusion, we should feel bound to regard such supposed ellipticity with the greatest suspicion, in the face of the result just stated.

The state of internal stress of an elastic sphere under tide-generating forces is identical with that caused by ellipticity of figure.† Hence the investigation of § 5 gives the distribution of stress-difference caused in the earth by the moon's attraction. In Plate 19, fig. 1, the point called "the pole" is the point where the moon is in the zenith.

Computation shows that the stress-difference at the surface, due to the lunar tide-generating forces, is 16 grammes per square centimeter, and at the centre eight times as much. These stresses are considerable, although very small compared with those due to terrestrial inequalities, as will appear below.

In § 6 the stresses produced by harmonic inequalities of high orders are considered. This is in effect the case of a series of parallel mountains and valleys, corrugating a mean level surface with an infinite series of parallel ridges and furrows. In this case compressibility makes absolutely no difference in the result, as shown in § 10.

It is found that the stress-difference depends only on the depth below the mean surface, and is independent of the position of the point considered with regard to ridge and furrow; the direction of the stresses does however depend on this latter consideration.

In Plate 19, fig. 2, is shown the law by which the stress-difference increases and then diminishes as we go below the surface. The vertical ordinates of the curve indicate the relative magnitude of the stress-difference, and the horizontal ones the depth below the surface. The depth OL on the figure is equal to the distance between contiguous ridges, and the figure shows that the stress-difference is greatest at a depth equal to $\frac{1}{6 \cdot 28}$ of OL.

The greatest stress-difference depends merely on the height and density of the mountains, and the depth at which it is reached merely on the distance from ridge to ridge.

Numerical calculation shows that if we suppose a series of mountains, whose crests are 4000 meters or about 13,000 feet above the intermediate valley-bottoms, formed of rock of specific gravity 2·8, then the maximum stress-difference is 2·6 tons per square inch (about the tenacity of cast tin); also if the mountain chains are 314 miles apart the maximum stress-difference is reached at 50 miles below the mean surface.

* The data for the calculation are: Ratio of terrestrial radius to Martian radius 1·878. Ratio of Martian mass to terrestrial mass ·1020. Whence ratio of Martian gravity to terrestrial gravity ·3596. Central stress-difference, due to ellipticity e, $996e$ tons per square inch. "Homogeneous" ellipticity of Mars $\frac{1}{166}$; and $\frac{996}{166}$ equal to 6.

† This is subject to certain qualifications noticed in § 5.

It may be necessary to warn the geologist that this investigation is approximate in a certain sense, for the results do not give the state of stress actually within the mountain prominences or near the surface in the valley-bottoms. The solution will however be very nearly accurate at some five or six miles below the valley-bottoms. The solution shows that the stress-difference is *nil* at the mean surface, but it is obvious that both the mountain masses and the valley-bottoms are in some state of stress.

The mathematician will easily see that this imperfection arises, because the problem really treated is that of an infinite elastic plane, subjected to simple harmonic tractions and pressures.

To find the state of stress actually within the mountain masses would probably be difficult.

The maximum stress-difference just found for the mountains and valleys obviously cannot be so great as that at the base of a vertical column of this rock, which has a section of a square inch and is 4000 meters high. The weight of such a column is 7·1 tons, and therefore the stress-difference at the base would be 7·1 tons per square inch. The maximum stress-difference computed above is 2·6, which is about three-eighths of 7·1 tons per square inch. Thus the support of the contiguous masses of rock, in the case just considered, serves as a relief to the rock to the extent of about five-eighths of the greatest possible stress-difference. This computation also gives a rough estimate of the stress-differences which must exist if the crust of the earth be thin. It is shown below that there is reason to suppose that the height from the crest to the bottom of the depression in such large undulations as those formed by Africa and America is about 6000 meters. The weight of a similar column 6000 meters high is nearly 11 tons.

In § 7 I take the cases of the even zonal harmonics from the 2nd to the 12th, but for all except the 2nd harmonic only the equatorial region of the sphere is considered.

Plate 19, fig. 3, shows an exaggerated outline of the equatorial portion of the inequalities; it only extends far enough to show half of the most southerly depression, even for the 12th harmonic. It did not seem worth while to trace the surfaces of equal stress-difference throughout the spheroid, but the laborious computations are carried far enough to show that these surfaces must be approximately parallel to the surface of the mean sphere. It is accordingly sufficient to find the law for the variation of stress-difference immediately underneath the equatorial belt of elevation. It requires comparatively little computation to obtain the results numerically, and the results of the computation are exhibited graphically in Plate 20, fig. 4.

Table V. (*b*), § 7, gives the maximum stress-differences, resulting from these several inequalities, computed under conditions adequately noted in the table itself. It will be convenient to postpone the discussion of the results.

In § 8 I build up out of these six harmonics an isolated equatorial continent. The nature of the elevation is exhibited in Plate 20, fig. 5, in the curve marked "representation;" no notice need be now taken of the dotted curve. This curve exhibits a belt of elevation of about 15° of latitude in semi-breadth, and the rest of the spheroid is approximately spherical. This kind of elevation requires the 2nd as one of its harmonic constituents, and this harmonic means ellipticity of the whole globe. Now it may perhaps be fairly contended that on the earth we have no such continent as would require a perceptible 2nd harmonic constituent. I therefore give in Plate 20, fig. 5, a second curve which represents an equatorial belt of elevation counterbalanced by a pair of polar continents in such a manner that there is no second harmonic constituent.

I have not attempted to trace the curves of equal stress-difference arising from these two kinds of elevation, but I believe that they will consist of a series of much elongated ovals, whose longer sides are approximately parallel with the surface of the globe, drawn about the maximum point in the interior of the sphere at the equator. The surfaces of equal stress-difference in the solid figure will thus be a number of flattened tubular surfaces one within the other.

At the equator however the law of variation of stress-difference is easy to evaluate, and Plate 20, fig. 6, shows the results graphically, the vertical ordinates representing stress-difference and the horizontal the depths below the surface. The upper curve in Plate 20, fig. 6, corresponds with the "representation curve" of Plate 20, fig. 5, and the lower curve with the case where there is no 2nd harmonic constituent.

The central stress-difference, which may be observed in the upper curve, results entirely from the presence of the 2nd harmonic constituent in the corresponding equatorial belt of elevation.

The maximum stress-differences in these two cases occur at about 660 and 590 miles from the surface respectively.

We now come to perhaps the most difficult question with regard to the whole subject—namely, how to apply these results most justly to the case of the earth.

The question to a great extent turns on the magnitude and extent of the superficial inequalities in the earth. As the investigation deals with the larger inequalities, it will be proper to suppose the more accentuated features of ridges, peaks, and holes to be smoothed out.

The stresses caused in the earth by deficiency of matter over the sea beds are the same as though the seas were replaced by a layer of rock, having everywhere a thickness of about $\frac{1 \cdot 02}{2 \cdot 75}$ or nearly $\frac{4}{11}$ of the actual depths of sea.

The surface being partially smoothed and dried in this manner, we require to find an ellipsoid of revolution which shall intersect the corrugations in such a manner that the total volume above it shall be equal to the total volume below it.

Such a spheroid may be assumed to be the figure of equilibrium appropriate to the earth's diurnal rotation; if it departs from the equilibrium form by even a little, then we shall much underestimate the stress in the earth's interior by supposing it to be a form of equilibrium.

Professor BRUNS has introduced the term "geoid" to express any one of the "level" surfaces in the neighbourhood of the earth's surface, and he endeavours to form an estimate of the departure of the continental masses and sea-bottoms from some mean geoid.* From the geodesic point of view the conception is valuable, but such an estimate is scarcely what we require in the present case. The mean geoid itself will necessarily partake of the contortions of the solid earth's surface, even apart from disturbances caused by local inequalities of density, and thus it cannot be a figure of equilibrium.

Thus, even if we were to suppose that the solid earth were everywhere coincident with a geoid—which is far from being the case—a state of stress would still be produced in the interior of the earth.

An example of this sort of consideration is afforded by the geodesic results arrived at by Colonel CLARKE, R.E.,† who finds that the ellipsoid which best satisfies geodesic measurement, has three unequal axes, and that one equatorial semi-axis is 1524 feet longer than the other. Now such an ellipsoid as this, although not exactly one of BRUNS' geoids, must be more nearly so than any spheroid of revolution; and yet this inequality (if really existent, and Colonel CLARKE's own words do not express any very great confidence) must produce stress in the earth. Colonel CLARKE's results show an ellipticity of the equator equal to $\frac{1}{13731}$, and this in the homogeneous elastic earth will be about equivalent to ellipticity $\frac{1}{27000}$; such ellipticity would produce a central stress-difference of $\frac{7698}{27000}$ or nearly one-third of a British ton per square inch.

From this discussion it may, I think, be fairly concluded that if we assume the sea-level as being the figure of equilibrium and estimate the departures therefrom, we shall be well within the mark.

The average height of the continents is about 350 meters (1150 feet), and the average depth of the great oceans is in round numbers 5000 meters (16,000 feet); but the latter datum is open to much uncertainty.‡ When the sea is solidified into rock the 5000 meters of depth is reduced to 3200 meters below the actual sea-level. Thus the average effective depression of sea-bed is about nine times as great as the average height of the land. I shall take it as exactly nine times as great, and put the depth as 3150 meters; but it is of course to be admitted that perhaps eight and perhaps ten might be more correct factors.

In the analytical investigation of this paper the outlines of the vertical section of the continents and depressions are always sweeping curves of the harmonic type, and the magnitude of the elevations and depressions are estimated by the greatest heights and depths, measured from a mean surface which equally divides the two.

We have already supposed the outlines of continents and sea-beds to have been smoothed down into sweeping curves, which we may take as being, roughly speaking, of the harmonic type. The smoothing will have left the averages unaffected.

* 'Die Figur der Erde.' Von Dr. H. BRUNS. Berlin: STANKIEWICZ, 1878.

† Phil. Mag., Aug., 1878.

‡ In a previous paper, "Geological Changes, &c.," Phil. Trans., Vol. 167, Part I., p. 295, I have endeavoured to discuss this subject, and references to a few authorities will be found there.

The averages are not however estimated from a mean spheroidal surface, but from one which is far distant from the mean.

The questions now to be determined are as follows:—What is the proper greatest height and depression, estimated from a mean spheroid, which will bring out the above averages estimated from present sea-level, and what is the position of the mean spheroid with reference to the sea-level.

From the solution of the problem considered in the note below,* it appears that, if

* Conceive a series of straight harmonic undulations corrugating a mean horizontal surface, and suppose them to be flooded with water. This will represent fairly well the undulations on the dried earth, and the water-level will represent the sea-level.

Suppose that the average heights and depths of the parts above and below water are known, and that it is required to find the position of the mean horizontal surface with reference to the water-level, and the height of the undulations measured from that mean surface.

Take an origin of coordinates in the water-level, the axis of x in the water-level and perpendicular to the undulations, and the axis of y measured upwards.

Let

$$y=h(\cos x-\cos a)$$

be the equation to the undulations.

The average height of the dry parts is clearly $\frac{1}{2a}\int_{-a}^{+a} y\,dx$ or $\frac{h}{a}(\sin a-a\cos a)$. Similarly the average depth below water is $\frac{h}{\pi-a}[\sin(\pi-a)-(\pi-a)\cos(\pi-a)]$ or $\frac{h}{\pi-a}\sin a+(\pi-a)\cos a$

If the latter average be p times as great as the former

$$ph\cos a\left(\frac{1}{a}\tan a-1\right)=h\cos a\left(\frac{1}{\pi-a}\tan a+1\right)$$

This is an equation for determining a.

Now I find that $a=34°\ 30'$ gives $p=8{\cdot}983$, which corresponds very nearly with $p=9$ of the text above.

This value of a corresponds with an average equal to $\cdot 1165h$ for the height above water, and $1{\cdot}0469h$ for the depth below water. Now if we put

$$1{\cdot}0469h=3150 \text{ meters}$$

$$\text{which gives}\quad \cdot 1165h=350 \text{ meters very nearly,}$$

we have $h=3009$ meters.

The depth below water-level of the mean level is $h\cos 34°\ 30'$ or 2480 meters.

The greatest height of the dry part above the water-level is $3009-2480$ or 429 meters, and the greatest depth of the submerged part below water-level is $3009+2480$ or 5489 meters.

[After the proof-sheets of this paper had been corrected, Professor STOKES pointed out to me that, according to RIGAUD (Cam. Phil. Soc., vol. 6), the area of land is about four-fifteenths of the whole area of the earth's surface. Now, in the ideal undulations we are here considering the area above water is about one-tenth of the whole area; hence in this respect the analogy is not satisfactory between these undulations and the terrestrial continents. If I have not considerably over-estimated the average depth of the sea (and I do not think that I have done so), the discrepancy must arise from the fact that actual continents and sea-beds do not present in section curves which conform to the harmonic type; there must also be a difference between corrugated spherical and plane surfaces.

The geological denudation of the land must, to some extent, render our continents flat-topped.—*Added May 4th*, 1882.]

the continents and sea-beds have sections which are harmonic curves, then if we take,—

The mean level bisecting elevations and depressions as 2480 meters (8150 feet) below the sea-level, and the greatest elevation and depression from that mean level as 3009 meters (9840 feet), it results that the *average* height of the land above sea-level is 350 meters and the *average* depression of dried sea-bed is 3150 meters.

It thus appears that 3000 meters would be a proper greatest elevation and depression to assume for the harmonic analysis of this paper, if the earth were homogeneous. But as the density of superficial rocks is only a half of the mean density of the earth, I shall take 1500 meters as the greatest elevation and depression from the mean equilibrium spheroid of revolution.

It is proper here to note that the height of the undulations of elevation and depression in the zonal harmonic inequalities is considerably greater towards the poles than it is about the equator; it might therefore be maintained that by making 1500 meters the equatorial height, we are taking too high an estimate. But the state of stress caused in the sphere at any point depends very much more on the height of the inequality in the neighbourhood of a superficial point immediately over the point considered, than it does on the inequalities in remote parts of the sphere.

Now in all the inequalities, except the 2nd harmonic, I have considered the state of stress in the equatorial region, and it will therefore I think be proper to adhere to the 1500 meters for the greatest height and depression.

We have next to consider, what order of harmonic inequalities is most nearly analogous to the great terrestrial continents and oceans. The most obvious case to take is that of the two Americas and Africa with Europe. The average longitude of the Americas is between 60° and 80° W., and the average longitude of Africa is about 25° E., hence there is a difference of longitude of about a right-angle between the two masses. These two great continents would be more nearly represented by an harmonic of the sectorial class,* rather than by a zonal harmonic, nevertheless I think the solution for the zonal harmonic will be adequate for the present purpose.

Now it has been explained above that the harmonic of the fourth order represents an equatorial continent and a pair of polar continents. In the case of the 4th harmonic therefore there is a right angle of a great circle between contiguous continents. We may conclude from this that the large terrestrial inequalities are about equivalent to the harmonic of the fourth order.

Table V. (*b*), § 7, gives the maximum stress-differences under the centre of the equatorial elevation of the several zonal harmonics, the height of each being 1500 meters.

* The sectorial harmonic of the fourth order $\sin^4\theta\cos 4\phi$ would well represent these two great continents. It would fairly represent China and Australia; but would annihilate the Himalayan plateau, and place another great continent in mid-Pacific. It is not at all difficult to find the stress-difference under the centre of a sectorial inequality, but to find it generally involves the solution of a cubic equation.

The point at which this maximum is reached is given in each case, and Plate 20, fig. 4, illustrates graphically the law of variation of stress-difference.

The second harmonic cannot be said to represent a continent, and the table shows that in each of the other cases the maximum stress-difference is very nearly 4 tons per square inch. The depths of the maximum point are of course very different in each case.

We have concluded above that Africa and America are about equivalent to an harmonic of the fourth order, hence it may be concluded that the stress-difference under those continents is at a maximum at more than 1100 miles from the earth's surface, and there amounts to about 4 tons per square inch. A comparison with Table VII. shows that marble would break under this stress, but that *strong* granite would stand.

The case of the isolated continent investigated in § 8 appeared likely to prove the most interesting one, for the purpose of application to the case of the earth. But unfortunately I have found it difficult to arrive at a satisfactory conclusion as to the proper height to attribute to the continent.

The average height of the American continent is about 1100 feet above the sea, and the average depth of the Pacific Ocean about 15,000 feet. If the water of the Pacific be congealed into rock, it will have an effective depth of 10,000 feet. The greatest height of the American continent above the bed of the dried Pacific when smoothed down must be fully 12,000 feet or 3700 meters. The height of the great central Asian plateau above the average bed of the southern ocean (after drying) must be considerably more than this.

Now in the application to the homogeneous planet the heights are to be halved to allow for the smaller density of surface rock.

I therefore take 2000 meters as the height of the top of the equatorial table-land above the remaining approximately spherical portion of the sphere.

The investigation of § 8 then shows that the equatorial table-land will give rise to a stress-difference of 4·1 tons per square inch at a depth of 660 miles; and that the equatorial table-land counterbalanced by the pair of polar continents (the second harmonic constituent being absent) gives a stress difference of about 3·8 tons per square inch at a depth of 590 miles.

This estimate of stress-difference agrees in amount, with singular exactness, with that just found from the case of the 4th zonal harmonic, but the maximum is reached 400 or 500 miles nearer to the earth's surface.

I think there can be no doubt but that there *are* terrestrial inequalities of much greater breadth than that of my isolated continent; thus this investigation for the isolated continent will give a position for the maximum stress-difference too near the surface to correspond with the largest continents. On the other hand, I do not feel at all sure that I have not considerably underestimated the height of such a comparatively narrow plateau.

In the present paper it has been impossible to take any notice of the stresses produced by the most fundamental inequality on the earth's surface, because it depends essentially on heterogeneity of density.

It is well known that the earth may be divided into two hemispheres, one of which consists almost entirely of land, and the other of sea. If the south of England be taken as the pole of a hemisphere, it will be found that almost the whole of the land, excepting Australia, lies in that hemisphere, whilst the antipodal hemisphere consists almost entirely of sea. This proves that the centre of gravity of the earth's mass is more remote from England, than the centre of figure of the solid globe.

A deformation of this kind is expressed by a surface harmonic of the first order, for such an harmonic is equivalent to a small displacement of the sphere as a whole, without true deformation. Now if we consider the surface forces produced by such a deformation in a homogeneous sphere, we find of course that there is an unbalanced resultant force acting on the whole sphere in the direction diametrically opposed to that of the equivalent displacement of the whole sphere.

The fact that in the homogeneous sphere such an unbalanced force exists shows that in this case the problem is meaningless; it is in fact merely equivalent to a mischoice in the origin for the coordinates. But in the case of the earth such an inequality does exist, and the force referred to must of course be counterbalanced somehow. The balance can only be maintained by inequalities of density, which are necessarily unknown. The problem therefore apparently eludes mathematical treatment.

It is certain that so wide-spreading an inequality, even if not great in amount, must produce great stress within the globe. And just as the 2nd harmonic produces a more even distribution of stress than the 4th, so it is likely that the first would produce a more even distribution than the 2nd.

It is difficult to avoid the conclusion that the whole of the solid portion of the earth is in a sensible state of stress.

I would not however lay very much emphasis on this point, because we are in such complete ignorance as to the manner in which the equilibrium of the solid part of the earth is maintained.

From this discussion it appears that if the earth be solid throughout, then at a thousand miles from the surface the material must be as strong as granite. If it be fluid or gaseous inside, and the crust a thousand miles thick that crust must be stronger than granite, and if only two or three hundred miles in thickness much stronger than granite. This conclusion is obviously strongly confirmatory of Sir William Thomson's view that the earth is solid throughout.

Fig. 1.

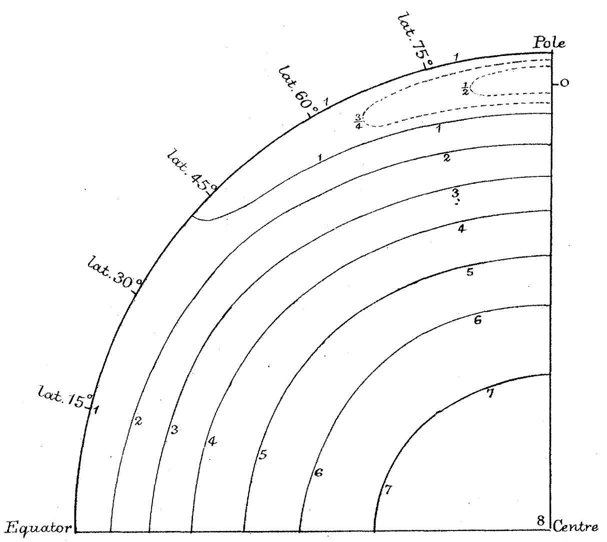

Diagram showing curves of equal stress-difference due to the weight of 2nd harmonic inequalities or to tide-generating force.

Fig. 3.

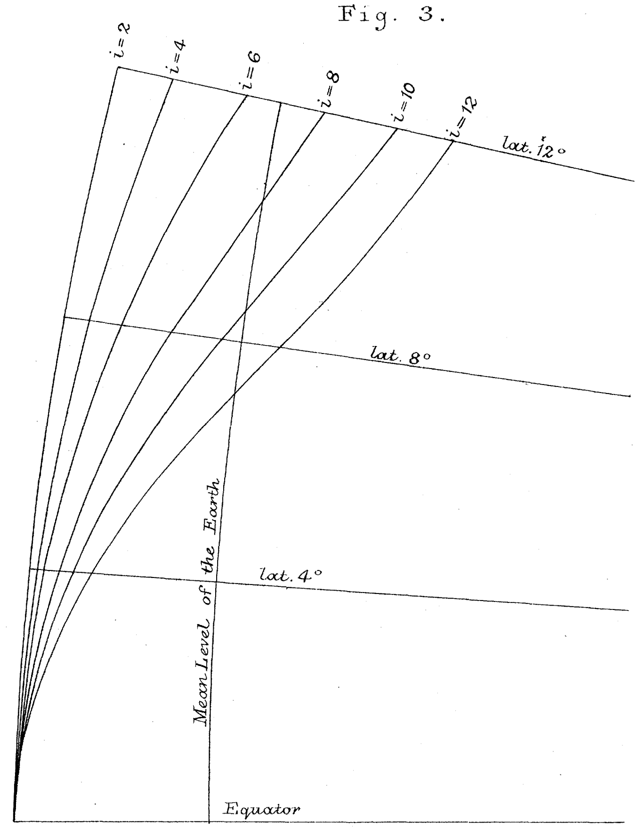

Diagram showing the profile of the even harmonics near the Equator. Radius of sphere 18 inches.

Fig. 2.

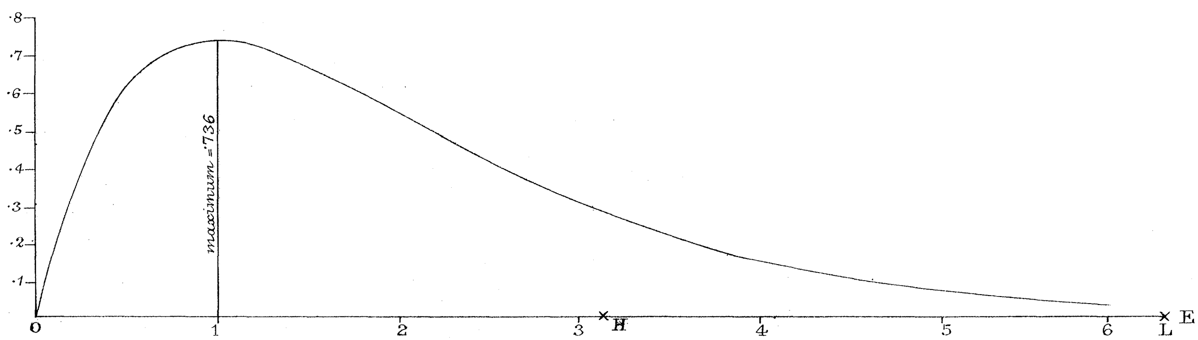

Diagram showing the difference between the principal stresses due to harmonic mountains and vallies on a horizontal plane.

Fig. 4.

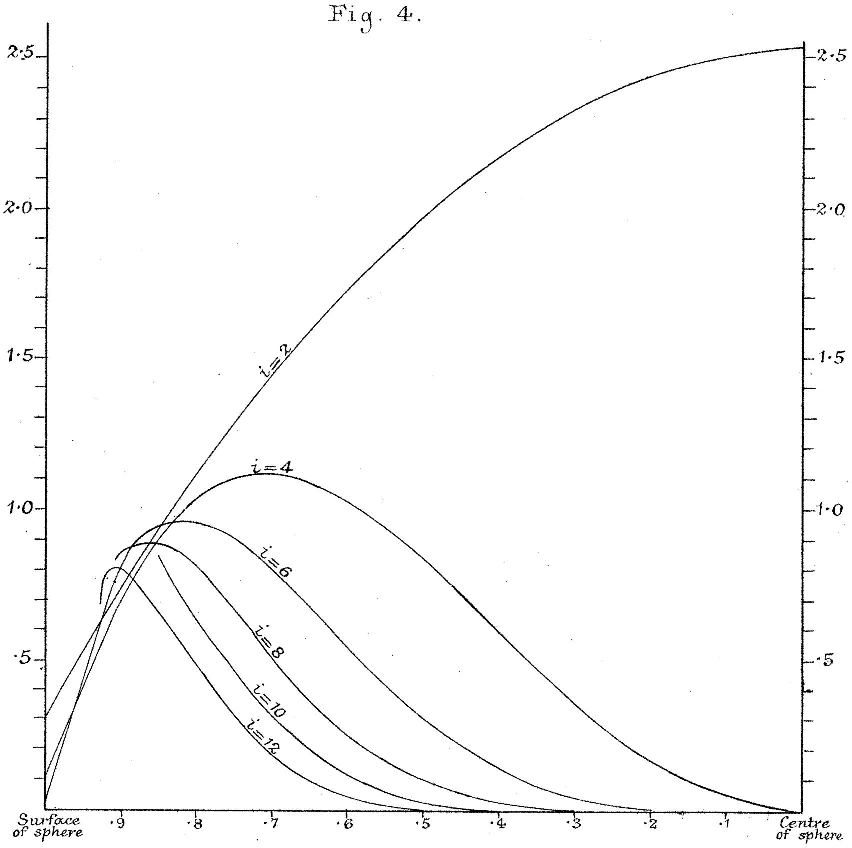

Diagram showing the difference of the principal stresses at the equator, due to inequalities represented by the even zonal harmonics.

Fig. 5.

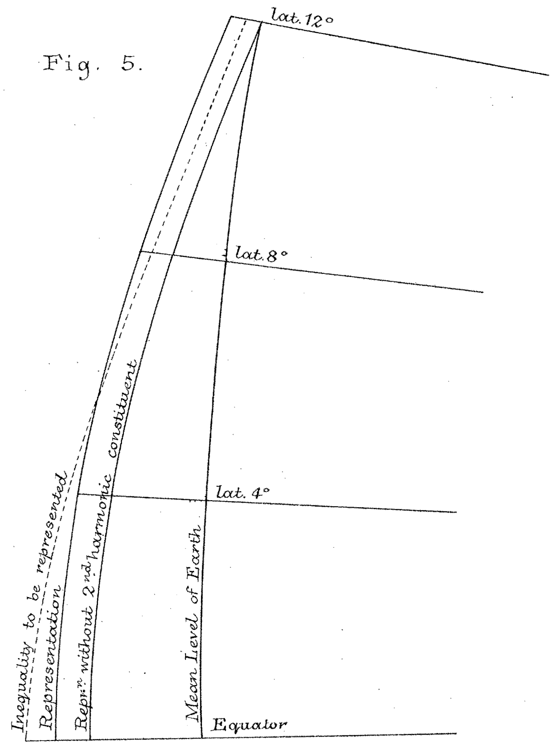

Diagram showing the profile of isolated equatorial continents.
Radius of sphere 18 inches.

Fig. 6.

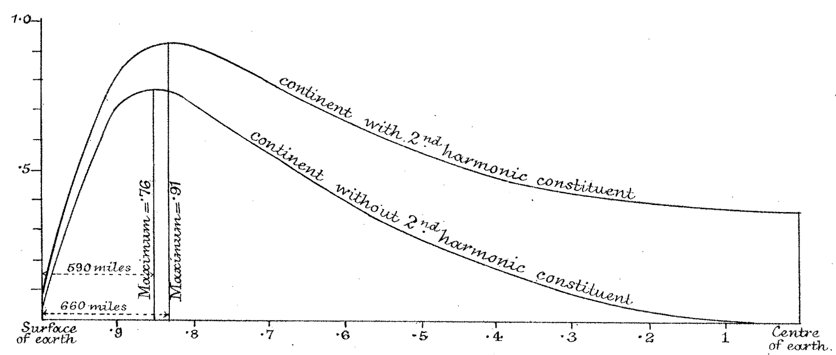

Diagram showing the difference of principal stresses at the equator due to isolated equatorial continents.

West, Newman & Co. lith.